RIGHT ORDERS IN FULL LINEAR RINGS

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by

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PREFACE

The Goldie theorems on prime and semi-prime rings have added greatly to our knowledge of non-commutative Noetherian rings, and have stimulated much further research. One way in which we can attempt to generalize the Goldie theorems is by studying right orders in rings which may have nonzero radical but which still possess a good deal of finiteness (for example, Artinian rings in general, quasi-Frobenius rings, semi-perfect rings). A number of people have studied these rings in the last decade and a host of interesting results have been obtained. In the other direction, we can relax the finiteness requirement and study right orders in rings with zero (Jacobson) radical, for example, regular right self-injective rings. The rings studied here, right orders in (left) full linear rings, fall into the latter category. The problem of characterizing such rings was posed by Faith [1], problem 12, p.129. Since a regular right self-injective ring whose socle is large as a one-sided ideal is a direct product of full linear rings (see Chase and Faith [1], Dlab [1], Johnson [4]), it is clear how one could approach the study of right orders in these rings from our study.

Our approach to the problem at hand is via the notion of a right quotient ring in the sense of R.E. Johnson [1]. (Nowadays, the term "quotient ring" often conveys a much broader concept than the one we consider here.) The Johnson maximal right quotient ring is an excellent setting in which to study right orders in simple Artinian rings. For right orders in infinite dimensional full linear rings, however, the flat epimorphic hull of a ring, as advocated very recently by Popescu and Spircu [1], Findlay [1], and Morita [2], may prove to be a more appropriate setting.

Chapter I is an introductory chapter and contains a summary of known results needed in the sequel. One of the principal results in chapter II is that right orders in full linear rings of countable dimension must be prime rings, whereas in the uncountable case this need not be so. Chapter III is a study of intrinsic extensions of prime rings. This study was required by the condition that regular elements of a right order R in a full linear ring Q be units in Q, since this actually implies Q is left intrinsic over R if Q is infinite dimensional. The main goal of chapter IV may perhaps be best described, in the recent terminology of Findlay [1], as finding suitable conditions to ensure that a ring will have a full linear ring as its left-flat epimorphic hull. In chapter V, we offer some necessary and sufficient conditions for a ring to be a right order in an infinite dimensional full linear ring, but by no means can they be considered as the final word on the subject.

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CHAPTER I

DEFINITIONS AND PRELIMINARY RESULTS

Our basic source of reference throughout this work will be Faith's lecture notes [1]. The terminology and notation will, for the most part, be taken from there. A ring is not required to possess an identity element, although associativity is always assumed. The unqualified word ideal means two-sided ideal.

The present chapter is divided into two sections. §1 consists of a brief summary of the basic notions involved in the study of (R.E. Johnson) quotient rings. With the exception of the notion of uniform dimension of modules which are not necessarily finite dimensional, an account of which appears at the end of §1, Faith's notes give an excellent coverage of these ideas. In §2, we outline our approach to the study of right orders in left full linear rings and also collect, for the purpose of easy reference, some well known results on full linear rings.

SECTION 1

A right (resp. left) module M over a ring R will be denoted by M_R (resp. ${}_RM$). For the concepts of *large submodule, singular submodule, injective module, injective hull* and related concepts, the reader is referred to Faith [1]. We shall write $(M \nabla N)_R$ to indicate that N_R is a large submodule of M_R. We also describe this situation by saying that N_R is *essential in* M_R or that M_R is an *essential extension* of N_R. The lattice of all submodules (resp. large submodules) of a module M_R is denoted by $L(M_R)$ (resp. $L^{\Delta}(M_R)$). The singular submodule of M_R is denoted by $Z(M_R)$. In the case of the module R_R, R a ring, we write $L_r^{\Delta}(R)$ and $Z_r(R)$ in place of $L^{\Delta}(R_R)$ and $Z(R_R)$. If $M_1,\dots,M_n$ are submodules of M_R we shall indicate that their sum is direct by writing the sum as $M_1 \dotplus \dots \dotplus M_n$. For a subset X of a module M_R, the annihilator of X in R will be written as $r(X,R)$, or simply as X^r if there is no confusion as to where the annihilator is taken from. In any case the context should make it clear.

A submodule N_R of M_R is called a *closed submodule* of M_R if N_R has no proper essential extension within M_R. In case $Z(M_R) = 0$, a closure operation s can be defined on $L(M_R)$ as follows: for $N \in L(M_R)$, $N^s = \{x \in M\colon xI \subseteq N$ for some $I \in L_r^{\Delta}(R)\}$. Then N is a closed submodule of M if and only if $N = N^s$ (see Johnson [4], Faith [1] pp. 15, 61). In general, for a submodule N_R of M_R, N^s is the unique maximal essential extension of N_R in M_R. The set of all closed submodules of M_R is a complete complemented modular lattice under set-theoretic intersection and a union operation $\vee$ given by $A \vee B = (A + B)^s$, A and B closed submodules. We denote this lattice by $L^s(M_R)$.

Needless to say, the terms *closed right ideal, large right ideal* etc. of a ring R, refer to the module R_R. The notation to be used when we are working with left modules should be clear. For example, $Z_\ell(R)$ denotes the left singular ideal of a ring R.

The following proposition, due to R.E. Johnson [2], is of fundamental importance in studying essential extensions. Its proof can also be found in Faith [1] p.61.

<u>PROPOSITION 1.1.1</u> Let M_R have $Z(M_R) = 0$ and suppose N_R is essential in M_R. Then $L^s(M_R)$ is isomorphic to $L^s(N_R)$ under the contraction map $A \to A \cap N$, $A \in L^s(M_R)$.

<u>PROPOSITION 1.1.2</u> Let R be a ring with zero right singular ideal. Then right annihilator ideals of R are closed right ideals. Moreover, if I and J are right ideals of R with J_R essential in I_R, then $I^\ell = J^\ell$.

The proof is quite straightforward.

If R is a subring of a ring S, then following R.E. Johnson [1] we shall call S a *right quotient ring* of R if S_R is an essential extension of R_R, where the module S_R is defined in the natural way. Johnson in [1] showed that a ring R possesses a right quotient ring which is a (von Neumann) regular ring if and only if R has zero right singular ideal. In this case R has a unique (up to isomorphism over R) maximal right quotient ring, which we denote as $\hat{R}$. Johnson and Wong in [1] showed that $\hat{R}$ is in fact the injective hull of R_R supplied with a module-preserving ring structure, and that $\hat{R}$ is a regular right self-injective ring (necessarily with identity A very natural realization of $\hat{R}$ is the ring $\Lambda = \mathrm{Hom}_R(E,E)$, where E_R is the injective hull of R_R. R is embedded in Λ under the map $r \to r^*$, where r^* is the unique element of Λ which induces the map $a \to ra$, $a \in R$. Faith [1], §8, has all the details. See also Michler [1]. Johnson's original construction of $\hat{R}$ is given in Johnson [1]. In the sequel we abbreviate maximal right quotient ring to MRQ ring.

Historical remark. Utumi [1] constructed maximal right quotient rings for rings with zero left annihilator. His definition of right quotient ring is a little more restrictive than Johnson's, although they coincide in the case of rings with zero right singular ideal. Findlay and Lambek in [1], [2] then extended Utumi's construction to arbitrary rings. It was Lambek [1] who discovered that for a ring R with identity, the Utumi maximal right quotient ring of R can be realized as the bicommutator (= double centralizer) of the injective hull of R_R. More recently, it has been shown that certain other right quotient rings of R (in Utumi's sense) can be obtained as bicommutators of appropriate faithful injective modules V_R. (This is not true, in general, for every right quotient ring of R. See Fuller [1], remark (b), p.662.) Morita [1], for example, has shown this to be the case for a left-flat epimorphic extension of R (see chapter IV, §4) and has given a form for V_R in this case. Beachy [1] has some interesting generalizations to the case of right quotient rings of R which can be obtained as bicommutators of suitable fully divisible right R-modules.

The proof of the following important result can be found in Johnson [4] and in Faith [1] p.70.

<u>PROPOSITION 1.1.3</u> Let R be a ring with zero right singular ideal and let S be a right quotient ring of R.

The following statements hold.

(i) $Z(S_R) = Z_r(S) = 0$.

(ii) The closed submodules of S_R are the closed right ideals of S.

(iii) $L_r^s(S)$ is isomorphic to $L_r^s(R)$ under the contraction map $A \to A \cap R$, $A \in L_r^s(S)$.

A *regular element* of a ring R is an element c with the property that $\ell(c,R) = r(c,R) = 0$.

If R is a subring of a ring S with identity, then R is a *right order* in S if

(i) regular elements of R have two-sided inverses in S, and

(ii) the elements of S can be expressed in the form bc^{-1}, b and c in R with c a regular element of R.

In this case we refer to S as a *classical right quotient ring* of R.

Remarks. (1) If R has a classical right quotient ring then it is unique (up to isomorphism over R). The Ore condition (see, for example, Lambek [2] p.109) gives a necessary and sufficient condition for a ring to possess a classical right quotient ring but, in general, it is difficult to say whether or not a given ring satisfies it.

(2) Suppose R has $Z_r(R) = 0$. If R has a classical right quotient ring then its regular elements are units in its MRQ ring $\hat{R}$, and $S = \{bc^{-1}: b, \text{ regular } c \in R\}$ is a subring of $\hat{R}$. Clearly, S is then the classical right quotient ring of R. In general though, S is a proper subring of $\hat{R}$.

We now record some well known facts about regular rings and regular right self-injective rings. Proofs of these can be found in Faith [1] pp.42, 70.

<u>PROPOSITION 1.1.4</u> Let T be a regular ring with identity. The following statements hold.

(i) Each principal right ideal of T is generated by an idempotent of T and hence is a closed right ideal of T.

(ii) The sum and intersection of any two principal right ideals of T are again principal.

(iii) If in addition T is a right self-injective ring, then the closed right ideals of T are its principal right ideals.

A module M_R is called *uniform* if it is nonzero and each of its nonzero submodules is essential in M_R. M_R is said to be a finite dimensional module if M contains no infinite direct sum of nonzero submodules. Such a module contains uniform submodules and the length of any maximal direct sum of uniform submodules is an invariant called the (uniform) dimension of the module (see Goldie [1]). This notion of dimension is meaningful for any module M_R which contains uniform submodules; in fact, the technique used to establish this is essentially that used by Jacobson [1] p.62 for completely reducible modules. A complete account of this is contained in Miyashita [1] and Fort [1]. However, because of the frequent use of uniform dimension in subsequent work we shall give a brief treatment of it here, following Miyashita [1] proposition 1.8. We denote the cardinality of a set I by $|I|$.

PROPOSITION 1.1.5 Let M_R be a module which contains uniform submodules, and suppose $\{A_i\}_{i\in I}$, $\{B_j\}_{j\in J}$ are maximal families of independent uniform submodules of M. Then $|I| = |J|$.

Proof. First of all, let us make the following observation: suppose $\{C_k\}_{k\in K}$ is a maximal independent family of uniform submodules of M. If $\{C_k'\}_{k\in K}$ is a family of nonzero submodules of M with $C_k' \subseteq C_k$ for all $k \in K$, then $\{C_k'\}_{k\in K}$ is also a maximal independent family and $U \cap (\sum_{k\in K} C_k') \neq 0$ for each uniform submodule U of M. Now let $\{A_i\}_{i\in I}$ and $\{B_j\}_{j\in J}$ be given as in the statement of the proposition. Let $i_0 \in I$ and let $I_0 = \{i \in I: i \neq i_0\}$.

Since $A_{i_0} \cap [\sum_J (B_j \cap \sum_{I_0} A_i)] = 0$, there exists, by the above observation, $j_0 \in J$ such that $B_{j_0} \cap \sum_{I_0} A_i = 0$. By the maximality of the family $\{A_i\}_{i\in I}$, we have $(B_{j_0} + \sum_{I_0} A_i) \cap A_{i_0} \neq 0$. Hence $\{B_{j_0}\} \cup \{A_i\}_{i\in I_0}$ is also a maximal independent family of uniform submodules. If J is a finite set, say $J = \{j_1,\ldots,j_n\}$, and if $|J| < |I|$ then by repeating the above process we could find $\{i_1,\ldots,i_n\} \subseteq I$ such that $(B_{j_1} + \ldots + B_{j_n}) \cap \sum_{I'} A_i = 0$, where $I' = I \setminus \{i_1,\ldots,i_n\}$, which is clearly a contradiction. Hence if $|J|$ is finite then $|I| \leqslant |J|$.

On the other hand, suppose $|J|$ is infinite. For each $j \in J$ let $F(j)$ be the unique minimal subset of I such that $B_j \cap \sum_{F(j)} A_i \neq 0$. Trivially, $\bigcup_{j\in J} F(j) \subseteq I$. Let $i_0 \in I$ and choose $j_1,\ldots,j_n \in J$ such that $A_{i_0} \cap (B_{j_1} + \ldots + B_{j_n}) \neq 0$. Then $A_{i_0} \cap (\sum_{F(j_1)} A_i + \ldots + \sum_{F(j_n)} A_i) \neq 0$. Since $\{A_i\}_{i\in I}$ is an independent family, we must have $i_0 \in \bigcup_{k=1}^{n} F(j_k)$. Hence $I = \bigcup_{j\in J} F(j)$. Since J is an infinite set and each $F(j)$ is a finite set, we have $|I| \leqslant |J|$.

Thus regardless of the cardinality of J, we have shown that $|I| \leqslant |J|$. Likewise $|J| \leqslant |I|$ and hence $|I| = |J|$. This completes the proof of 1.1.5.

If a module M_R contains uniform submodules then Zorn's lemma ensures the existence of a maximal family of independent uniform submodules of M. Thus, in view of proposition 1.1.5, it makes sense to define the *uniform dimension* of M_R, denoted by dim M_R, to be the cardinal number of any such family.

SECTION 2

A *left full linear ring* Q is a ring which is isomorphic to the ring $\mathrm{Hom}_D(V,V)$ of all linear transformations of a right vector space V over a division ring D (transformations are written on the left of vectors). In the case where dim $V_D < \infty$ then Q is a simple Artinian ring. Goldie in [2] gave the following necessary and sufficient conditions for a ring R to be a right order in a simple Artinian ring.

(i) R is a prime ring.

(ii) R satisfies the maximum condition for right annihilator ideals.

(iii) R_R has finite dimension.

(A ring which satisfies (ii) and (iii) is sometimes referred to as a right Goldie ring.) Goldie also showed that (ii) could be replaced by: R has zero right singular ideal. Johnson and Wong in [2] and Johnson in [4] then showed how Goldie's theorem could be approached from the theory of quotient rings (in the sense of §1). Procesi in [1] offered yet another approach. The basic difference in each of these approaches is the method by which the classical right quotient ring of R is constructed. Goldie used the Ore construction, Johnson the MRQ ring. Procesi's clever construction involved constructing a vector space V_D such that R could be embedded in $\mathrm{Hom}_D(V,V)$ in a very natural way, and such that the latter is a right quotient ring of R. While we do not comment on the merits of each approach, or on subsequent approaches, of which many are variations of the above three, it does appear that if one is content to assume the known ideal structure of a simple Artinian ring, then Johnson's approach utilizes this best. For more detail on the various approaches to Goldie's theorem see Lesieur and Croisot [1], Procesi and Small [1], Johnson [6], Michler [1], Faith [1] §§9 and 10, Jacobson [1] appendix B, Lambek [2] chapter 4, §6, Koh and Luh [1], and Amitsur [1]. From the viewpoint of category theory, see Gabriel [1].

In our study of right orders in left full linear rings, we shall adopt the Johnson approach. This stems from the following observation: a ring R is a right order in a left full linear ring if and only if R has $Z_r(R) = 0$ and its MRQ ring is a left full linear ring and a classical right quotient ring of R. The problem then falls into two parts.

(1) Characterize rings R which have $Z_r(R) = 0$ and whose MRQ ring is a left full linear ring.

(2) Given a ring R which has a left full linear ring Q as a right quotient ring, find necessary and sufficient conditions for R to be a right order in Q.

(2) is the subject of chapters II-V. As regards (1), we shall take as part of the hypothesis that R has $Z_r(R) = 0$ (see remark (1) following 1.2.2). Then a solution to (1) is contained in Johnson [4]. (See Hutchinson [2] for an alternative solution.) We briefly sketch the details. Johnson in [3] calls a ring R (right) *irreducible* if $Z_r(R) = 0$ and if for each nonzero ideal A of R, $A \cap A^{\ell} = 0$ implies $A^{\ell} = 0$.

PROPOSITION 1.2.1 Let R be a ring with zero right singular ideal. Then R is an irreducible ring if and only if its MRQ ring is a prime ring.

Proof. Let S be the MRQ ring of R. It is not hard to see that S will be a prime ring if and only if it contains no nontrivial central idempotents. Now let us suppose R is an irreducible ring. Let e be a nonzero central idempotent of S and let $A = eS \cap R$. Then A is a nonzero ideal of R and $\ell(A,R) = S(1-e) \cap R = (1-e)S \cap R$. Hence $\ell(A,R) \cap A = 0$. Since R is irreducible this implies $(1-e)S \cap R = 0$. Since S is a right quotient ring of R we must have $(1-e)S = 0$, that is, $e = 1$. Hence S contains no proper central idempotents and therefore S is a prime ring.

Conversely, suppose S is a prime ring. Let A be a nonzero ideal of R and let us suppose $A^{\ell} \cap A = 0$. Then $A^{\ell} + A$ is a large right ideal of R (since A^{ℓ} is the unique complement of A). By propositions 1.1.3 and 1.1.4, there exists an idempotent e of S such that $(eS \nabla A)_R$. Then $\ell(A,R) = (1-e)S \cap R$. Moreover, since A^{ℓ} and A are ideals of R, we have $(1-e)Re(A^{\ell}+A) = 0$ and $eR(1-e)(A^{\ell}+A) = 0$. Hence, since $Z(S_R) = 0$, we have $(1-e)Re = eR(1-e) = 0$. Thus $re = er$ for all $r \in R$. Now let $x \in S$ and choose a large right ideal I of R such that $xI \subseteq R$. Then $(xe-ex)I = 0$ and again $Z(S_R) = 0$ requires $xe - ex = 0$, that is, $xe = ex$. Thus e is a nonzero central idempotent of S. Since S is a prime ring we must have $e = 1$. Hence $A^{\ell} = 0$ and it follows that R is an irreducible ring. This completes the proof of 1.2.1.

Remarks. (1) Johnson sometimes preferred the following equivalent definition of an irreducible ring R: R has $Z_r(R) = 0$ and the lattice $L_r^s(R)$ has a trivial centre, that is, 0 and R are the only elements in $L_r^s(R)$ with unique complements. With this definition, 1.2.1 follows easily from 1.1.3.

(2) A prime ring with zero right singular ideal is clearly an irreducible ring. The converse is false, as is illustrated for example by the ring of all 2×2 upper triangular matrices over a field. In general, an irreducible ring is prime if and only if it contains no nonzero nilpotent ideals (see lemma 2.2.1).

It is well known that a left full linear ring Q is a prime ring with nonzero socle, and that Q is a right self-injective ring (see Faith [1] p.44). The converse is also known to hold (see Lambek [2] p.65, lemma 2, and Faith [1] p.73). Thus, by 1.1.3, 1.1.4 and 1.2.1 we have

PROPOSITION 1.2.2 A ring Q is a left full linear ring if and only if Q is a prime right self-injective ring with nonzero socle. A ring R with zero right singular ideal has a left full linear ring as its MRQ ring if and only if R is an irreducible ring containing uniform right ideals.

Remarks. (1) It is not clear just when a ring has zero right singular ideal. It is apparently not known whether even a primitive ring (without socle) must have zero singular ideal (see Faith [1] p.128, problem 7). However, in the presence of some finiteness in a ring, one can say a little more. A.C. Mewborn has shown that the following condition on a ring R is sufficient to ensure that $Z_r(R) = 0$ (see Johnson [6] p.38): R has a maximal right annihilator ideal N which satisfies $N^{\ell\ell} = 0$. The ring R is then necessarily irreducible (Johnson [6] lemma 1). In particular, of course, a prime ring which contains a maximal right annihilator ideal must have zero right singular ideal. For right self-injective rings with identity, Utumi in [6] showed that the right singular ideal coincides with the Jacobson radical.

(2) Notice that if R is an irreducible ring and R contains uniform right ideals, then $L_r^s(R)$ is atomic, that is, each nonzero element of $L_r^s(R)$ contains a minimal nonzero element of $L_r^s(R)$. In fact, by 1.2.2, $L_r^s(R)$ is isomorphic to the lattice of all subspaces of a right vector space.

(3) Koh and Mewborn in [1] showed that for a prime ring R, R has $Z_r(R) = 0$ and contains uniform right ideals if and only if R contains a maximal right annihilator ideal and a maximal closed right ideal.

(4) Johnson in [3] gave the following interesting characterization of an irreducible ring R containing uniform right ideals: R possesses a faithful uniform module M_R with $Z(M_R) = 0$ (c.f. a primitive ring).

The remainder of this section is concerned with left full linear rings.

NOTATION. Throughout this work the sole use of the letter Q will be to denote a left full linear ring. For a right ideal I of Q, we abbreviate dim I_Q to dim I.

Notice that if $Q = \mathrm{Hom}_D(V,V)$, V_D a vector space, then dim Q = dim V_D. For we may take for V a minimal left ideal Qe of Q, e a primitive idempotent of Q, and for D the division ring eQe. Then Q is a right quotient ring of V so that by 1.1.3, dim Q = dim V_V = dim V_D. However one must be careful to distinguish between dim Q and dim ${}_QQ$. They are equal only in the case when dim Q is finite. For letting W = eQ, one sees that ${}_W(Q \nabla W)$ and hence dim ${}_QQ$ = dim ${}_WW$ = dim ${}_DW$. Let ${}_DV^* = \mathrm{Hom}_D(V,D)$. Then ${}_DW \cong {}_DV^*$ under the map $\phi: W \to V^*$, $(\phi w)v = wv$. Hence if dim $Q = \aleph \geq \aleph_0$, then dim ${}_QQ$ = dim ${}_DV^* = a^{\aleph}$ where $a = |D|$ (see Jacobson [1] p.68).

Also, unless dim Q is finite, Q is not a left self-injective ring (see Sandomierski [2]).

<u>PROPOSITION 1.2.3</u> Let I and J be closed right ideals of Q. Then

(i) $I_Q \cong J_Q$ if and only if dim I = dim J.

(ii) If $I \cap J = 0$ then dim $(I + J)$ = dim I + dim J.

<u>Proof</u>. (i) Firstly, observe that for a family $\{I_\alpha\}_{\alpha \in \Omega}$ of independent uniform submodules of I_Q, each I_α is a minimal right ideal of Q, and the family is maximal if and only if $\sum_\Omega I_\alpha$ is essential in I_Q. Since dim I = dim J there exist an index set Ω and families $\{I_\alpha\}_{\alpha \in \Omega}$, $\{J_\alpha\}_{\alpha \in \Omega}$ of independent minimal right ideals of Q such that $I \nabla (\sum_\Omega I_\alpha)$ and $J \nabla (\sum_\Omega J_\alpha)$ as Q-modules. Moreover, since any two minimal right ideals of a prime ring are isomorphic, we can choose for each $\alpha \in \Omega$ an isomorphism ϕ_α of I_α onto J_α. Then the map $\phi : \sum_\Omega I_\alpha \to \sum_\Omega J_\alpha$, given by $\phi(\Sigma a_\alpha) = \Sigma \phi_\alpha(a_\alpha)$, is an isomorphism of $\sum_\Omega I_\alpha$ onto $\sum_\Omega J_\alpha$. Now as J_Q is a direct summand of the injective module Q_Q (proposition 1.1.4), J_Q is an injective module. Hence ϕ can be extended to $\hat{\phi} : I \to J$. Since $\sum_\Omega I_\alpha$ is essential in I, $\hat{\phi}$ is a monomorphism. Furthermore, by the injectivity of $\hat{\phi}(I)$ (as a Q-module), $\hat{\phi}(I)$ essential in J_Q implies $J = \hat{\phi}(I)$. Thus $I_Q \cong J_Q$, as desired. The converse is immediate.

(ii) Let $\{I_\alpha\}_{\alpha \in \Omega}$ and $\{I_\gamma\}_{\gamma \in \Gamma}$ be families of independent minimal right ideals of Q such that $\sum_\Omega I_\alpha$ and $\sum_\Gamma I_\gamma$ are essential in I_Q and J_Q respectively. We can suppose $\Omega \cap \Gamma = \phi$. Then $\{I_\beta\}_{\beta \in \Omega \cup \Gamma}$ is an independent family and $\sum_{\Omega \cup \Gamma} I_\beta$ is essential in $I + J$. Hence $\dim (I + J) = |\Omega \cup \Gamma| = |\Omega| + |\Gamma| = \dim I + \dim J$.

The following proposition gives a complete description of the ideals of Q. Its proof can be found in Jacobson [1] p. 93.

<u>PROPOSITION 1.2.4</u> Suppose dim Q is infinite. Then the sets $\{x \in Q: \dim xQ < \aleph\}$, $\aleph$ a cardinal satisfying $\aleph_0 \leqslant \aleph \leqslant \dim Q$, are the only proper ideals of Q.

It has been known for a long time that the idempotent elements of Q are its building blocks. We conclude our preliminaries by recalling a couple of things one can do with idempotents.

<u>PROPOSITION 1.2.5</u> Providing Q is not a division ring, Q is generated (as a ring) by its idempotents (and hence in turn by its nilpotent elements and its units). If dim Q is infinite, then Q is generated by the idempotents e for which $eQ \cong (1-e)Q \; (\cong Q)$.

<u>Proof</u>. In the case where $1 < \dim Q < \infty$, this is shown in lemma 3.2.2. If dim Q is infinite then Q is isomorphic to the ring of all 2×2 matrices over itself,

and it is a straightforward procedure showing that the ring T of all 2×2 matrices over any ring with identity is generated by the idempotents e for which $eT \cong (1-e)T$.

Remark. It is conceivably of interest to know what elements of Q can be expressed as products of idempotents. It can be shown that for $a \in Q$, $a \neq 1$, a can be expressed as a product of idempotents if and only if a satisfies one of the following two conditions:

(i) $\dim a^r = \dim (1-a)Q = \operatorname{codim}(a) \neq 0$.

(ii) a has the form $1+x$, $x \in$ socle Q, $1+x$ not a unit in Q.

(Here codim (a) is the dimension of any complement of aQ.) In particular, if Q is simple Artinian then any non-unit of Q is expressible as a product of idempotents.

We shall call a set $\{e_i\}_{i\in I}$ of nonzero orthogonal idempotents of Q a *complete set* if $\sum_I e_iQ \supseteq$ socle Q. Suppose $\{e_i\}_{i\in I}$ is such a set and that for each $i \in I$, $x_i \in Qe_i$ is given. Consider the map $\phi: \sum_I e_iQ \to Q$ given by $\phi(\Sigma a_i) = \Sigma x_i a_i$. Since Q_Q is injective, there exists $x \in Q$ such that $\phi(y) = xy$ for all $y \in \sum_I e_iQ$. Since socle $Q \subseteq \sum_I e_iQ$, x is unique. Clearly, $xe_i = x_i$ for all $i \in I$. Thus ${}_QQ$ is isomorphic to $\prod_{i\in I} Qe_i$ under the map $x \to (xe_i)$. Our final proposition summarizes this observation.

PROPOSITION 1.2.6 If $\{e_i\}_{i\in I}$ is a complete set of nonzero orthogonal idempotents of Q, then ${}_QQ$ is isomorphic to $\prod_{i\in I} Qe_i$ under the map $x \to (xe_i)$.

Remarks. (1) One can think of 1.2.6 as a generalized left Peirce decomposition. If dim Q is infinite then the corresponding right-sided version of 1.2.6 fails in general, that is, given $x_i \in e_iQ$ for all $i \in I$, there need not exist $x \in Q$ with $e_ix = x_i$ for all $i \in I$ (think of infinite by infinite column-finite matrices).

(2) If dim Q is infinite then a maximal set of orthogonal idempotents of Q need not be a complete set. (Thus there is a slight error in Faith [1] p.121.)

(3) Suppose dim Q = $\aleph$. Then Q contains a complete set $\{e_i\}_{i\in I}$ of orthogonal primitive idempotents with $|I| = \aleph$. This is a straightforward consequence of the definition of dim Q and the injectivity of Q_Q. Conversely, of course, any such set has cardinality $\aleph$.

The papers [1] and [2] of Osofsky are a particularly good source of information for deeper results on full linear rings. For a study of prime rings with nonzero socle, see Jacobson [1] chapter IV. See also Wolfson's paper [1] on full linear rings. Utumi's papers [2], [3], [6] and [7] contain a wealth of information on the idempotents and structure of regular rings and self-injective rings.

CHAPTER II

RIGHT ORDERS WHICH ARE NOT PRIME RINGS

Recall that Q denotes a left full linear ring. It is well known that when dim Q is finite, a right order in Q must be a prime ring. The object of this chapter is to show that this is also the case when dim Q is countable, but no longer the case when dim Q is uncountable. More explicitly, we show in §1 that if $e_1,\ldots,e_n$ are nonzero orthogonal idempotents of Q with $e_1+\ldots+e_n = 1$, $n \geq 2$, then the ring $P = \sum_{i\leq j} e_iQe_j$ is a right order in Q if and only if $\dim e_1Q > \dim e_iQ$ for $i = 2,\ldots,n$ and $\dim e_nQ \geq \aleph_0$ (theorem 2.1.4). The Jacobson radical of P is, of course, nilpotent of index n. In §2, we show that if R is a right order in Q and if R contains a nilpotent ideal N of index $n > 1$, then R must appear as a subring of P for some P given as above (theorem 2.2.2). This enables us to conclude, for example, that a left and right order in Q must be a prime ring. Unfortunately, one can not say very much about the subrings e_iRe_i, $i = 1,\ldots,n$, even if N is a maximum nilpotent ideal, or at least not for the e_i, $i = 1,\ldots,n$, arising from our construction. Finally, we indicate how to construct right orders in Q which do not contain a maximum nilpotent ideal and whose Jacobson radical is not nil.

SECTION 1

For an idempotent $e \in Q$ and an element $x \in Q$, we shall occasionally write $x = \begin{pmatrix} \alpha & \beta \\ \delta & \gamma \end{pmatrix}$ to mean that α, β, γ, δ are the components of x (via the standard two-sided Peirce decomposition) in eQe, $eQ(1-e)$, $(1-e)Q(1-e)$ and $(1-e)Qe$ respectively, providing there is no confusion as to what the underlying idempotent e is. Components of sums and products can then be computed by the usual 2×2 matrix operations.

THEOREM 2.1.1 Suppose e and f are nonzero orthogonal idempotents of Q. Let $P = eQ + Qf$. Then P is a right order in Q if and only if $\dim eQ = \dim Q > \dim fQ \geq \aleph_0$ and $\dim fQ = \dim (1-e)Q$.

Remark. P is not a prime ring since it contains the nilpotent ideal $eQ(1-e) + (1-f)Qf$.

Proof. Suppose e and f satisfy: $\dim eQ > \dim fQ \geq \aleph_0$ and $\dim fQ = \dim (1-e)Q$. Let $\aleph = \dim Q$. Then by proposition 1.2.3, $\aleph = \dim eQ = \dim (1-f)Q$. Since P contains a nonzero right ideal and a nonzero left ideal of the prime ring Q, Q is both a left and right quotient ring of P. Thus regular elements of P will remain regular in Q and hence will be units in Q. Hence to show P is a right order in Q, it will suffice to show that for each $\delta \in (1-e)Q$ there is a regular element $c \in P$ such that $\delta c \in P$. Let $\delta \in (1-e)Q$ be given. As Q is a regular ring, we can find idempotents $e_1, e_2 \in Q$ such that $\delta^r \cap eQ = e_1Q$ and $eQ = e_1Q \dotplus e_2Q$. Now δ, under its left multiplication, acts as a monomorphism on e_2Q. Hence,

$$\dim e_2Q = \dim \delta e_2Q \leqslant \dim (1-e)Q < \aleph.$$

By proposition 1.2.3, $\dim e_1Q + \dim e_2Q = \dim eQ = \aleph$. Thus, as $\aleph$ is an infinite cardinal, we have $\dim e_1Q = \aleph = \dim (1-f)Q$.

Claim: There exist idempotents $f_1, f_2 \in Q$ such that $fQ = f_1Q \dotplus f_2Q$, $\dim f_1Q = \dim (1-e)Q$ and $\dim f_2Q = \dim e_2Q$. To verify this, choose a family $\{f_\alpha Q\}_{\alpha \in \Omega}$ of minimal right ideals of Q such that the sum $\sum_\Omega f_\alpha Q$ is direct and is an essential submodule of $(fQ)_Q$. By definition, $\dim fQ = |\Omega|$. Since $|\Omega|$ is an infinite cardinal and $|\Omega| \geqslant \dim e_2Q$, we can write $\Omega = \Omega_1 \cup \Omega_2$ where $\Omega_1 \cap \Omega_2 = \phi$, $|\Omega| = |\Omega_1|$ and $|\Omega_2| = \dim e_2Q$. Choose idempotents $f_1, f_2 \in Q$ such that $f_1Q = (\sum_{\Omega_1} f_\alpha Q)^s$ and $f_2Q = (\sum_{\Omega_2} f_\alpha Q)^s$, where s denotes closure in $L(Q_Q)$. Then we have

$fQ = f_1Q \dotplus f_2Q$, $\dim f_1Q = |\Omega_1| = \dim (1-e)Q$ and $\dim f_2Q = |\Omega_2| = \dim e_2Q$.

Proposition 1.2.3 now enables us to construct isomorphisms $\alpha\colon (1-f)Q \to e_1Q$, $\beta\colon f_2Q \to e_2Q$ and $\gamma\colon f_1Q \to (1-e)Q$. Let ϕ be the homomorphism of Q_Q which induces each of α, β and γ, that is, ϕ has the diagram

$$\begin{tikzcd}
 & (1-f)Q \arrow[r, "\alpha"] & e_1Q \\
fQ \left\{ \right. & f_2Q \arrow[r, "\beta"] & e_2Q \left. \right\} eQ \\
 & f_1Q \arrow[r, "\gamma"] & (1-e)Q
\end{tikzcd}$$

Let c be the element of Q whose left multiplication induces ϕ. Clearly c is a unit of Q. Since $c(1-f) \in eQ$, $c = c(1-f) + cf \in eQ + Qf$, that is, $c \in P$. Moreover, $c(1-f) \in e_1Q \subseteq \delta^r$ so that $\delta c(1-f) = 0$. Thus $\delta c \in P$ and this establishes the "if part" of the theorem.

Conversely, suppose P is a right order in Q. Then so also is $P_1 = eQ + Q(1-e)$. If $\dim (1-e)Q$ is finite, then $(1-e)Q(1-e)$ is simple Artinian. In this case, if c is a regular element of P_1 then c^{-1} is also in P_1. For let $c = \begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix}$ and $c^{-1} = \begin{pmatrix} \alpha_1 & \beta_1 \\ \delta_1 & \gamma_1 \end{pmatrix}$ where components are taken with respect to e. A simple calculation yields

$\gamma\gamma_1 = 1 - e$ and $\gamma\delta_1 = 0$. But γ right invertible in the Artinian ring $(1-e)Q(1-e)$ implies that γ is also left invertible, that is, $\gamma_1\gamma = 1-e$. Thus $\delta_1 = (1-e)\delta_1 = \gamma_1\gamma\delta_1 = 0$, that is, $c^{-1} \in P$. It follows that $P_1 = Q$ which contradicts $e \neq 0, 1$. Thus we must have $\dim (1-e)Q \geqslant \aleph_0$.

Suppose $\dim eQ \leqslant \dim (1-e)Q$. Then by proposition 1.2.3 there exists a monomorphism of eQ into $(1-e)Q$ and hence there exists $\delta \in (1-e)Qe$ such that $\delta^r \cap eQ = 0$. Since P_1 is a right order in Q, there exists a regular element $c \in P_1$ such that $\delta c \in P_1$. Now $\delta c \in P_1$ implies $(1-e)\delta ce = 0$, that is, $\delta ece = 0$. Since $\delta^r \cap eQ = 0$, we must have $ece = 0$. But $c \in P_1$ implies $ece = ce$ and hence $ce = 0$, which contradicts c being a regular element of P_1. From this contradiction, we conclude that $\dim eQ > \dim (1-e)Q$.

To complete the proof we must show that $\dim (1-e)Q = \dim fQ$. Since e and f are orthogonal, we have $\dim (1-e)Q \geqslant \dim fQ$. Suppose $\dim (1-e)Q > \dim fQ$. Let $M = \{x \in Q : \dim xQ < \dim (1-e)Q\}$. Then M is an ideal of Q by proposition 1.2.4. Let $\bar{Q} = Q/M$ and for a subset X of Q let $\bar{X}$ denote the image of X under the canonical map of Q onto $\bar{Q}$. Then $(\overline{1-e})\bar{P} = \bar{0}$ and $\overline{1-e} \neq \bar{0}$, which implies that P contains no regular elements. Hence we must have $\dim (1-e)Q = \dim fQ$. We are finished.

Taking $f = 1-e$ we obtain:

COROLLARY 2.1.2 Let e be an idempotent of Q, $e \neq 0, 1$, and let $P = eQ + Q(1-e)$. Then P is a right order in Q if and only if $\dim eQ = \dim Q > \dim (1-e)Q \geqslant \aleph_0$.

Remarks. (1) The only use we made of e and f being orthogonal in 2.1.1 was to ensure that $eQ + Qf$ would be a subring of Q. If we do not require e and f to be orthogonal, the ring $A = eQ + Qf + QfeQ$ will be a right order in Q provided that $\dim eQ > \dim fQ \geqslant \dim (1-e)Q \geqslant \aleph_0$. In this case, A is still a proper subring of Q if (and only if) $\dim feQ < \dim fQ$. But we get nothing new in the sense that there is an idempotent $h \in Q$, with h orthogonal to e, $\dim hQ = \dim (1-e)Q$ and such that $A = eQ + Qh + QfeQ$.

(2) Suppose e is an idempotent of Q such that $\dim eQ > \dim (1-e)Q \geqslant \aleph_0$. Let I be a nonzero ideal of Q and let $R = eQ + Q(1-e) + I$. Let $Y = eQ + I$. Then $R = \{x \in Q : xY \subseteq Y\}$. Since Q_Q is injective and $Y^{\ell} = 0$, $R \cong \mathrm{Hom}_Q(Y,Y) = \mathrm{Hom}_Y(Y,Y)$. Hence if $\dim Q$ is uncountable, Q contains a large right ideal whose endomorphism ring is a (proper) right order in Q. Can this happen if $\dim Q$ is countable? We shall return to this question

in chapter V, §2.

We next establish a natural generalization of corollary 2.1.2. A lemma is required.

LEMMA 2.1.3 Suppose e is an idempotent of Q such that $\dim eQ > \dim (1-e)Q \geqslant \aleph_0$. Let K be a right order in $(1-e)Q(1-e)$ and T a right order in eQe, and suppose ${}_T(eQe \nabla T)$. Let $N = eQ(1-e)$ and $R = T + N + K$. Then R is a right order in Q.

Proof. It is clear that R is a subring of Q. Let $P = eQ + Q(1-e)$. Let $x \in Q$ be given. By 2.1.2, there is a regular element $c \in P$ such that $xc \in P$. Taking components with respect to e, we can write $c = \begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix}$ and $xc = \begin{pmatrix} \alpha_1 & \beta_1 \\ 0 & \gamma_1 \end{pmatrix}$, say. Choose regular elements α_2, γ_2 of T and K respectively such that $\alpha\alpha_2$, $\alpha_1\alpha_2 \in T$ and $\gamma\gamma_2$, $\gamma_1\gamma_2 \in K$. Let $d = \alpha\alpha_2 + \beta\gamma_2 + \gamma\gamma_2$, that is, $d = \begin{pmatrix} \alpha\alpha_2 & \beta\gamma_2 \\ 0 & \gamma\gamma_2 \end{pmatrix}$. Then d is a regular element of R and $xd \in R$.

The proof will be complete if we can show that regular elements of R are units in Q. However it is immediate that ${}_T(eQe \nabla T)$ implies ${}_R(Q \nabla R)$, which, together with the fact that $(Q \nabla R)_R$, is sufficient to ensure that regular elements of R remain regular in Q.

<u>THEOREM 2.1.4</u> Suppose $e_1, \ldots, e_n$ are nonzero orthogonal idempotents of Q with $e_1 + \ldots + e_n = 1$, $n \geq 2$. Let $P = \sum_{i \leq j} e_i Q e_j$. Then P is a right order in Q if and only if $\dim e_1 Q = \dim Q > \dim e_i Q$ for $i = 2, \ldots, n$ and $\dim e_n Q \geq \aleph_0$.

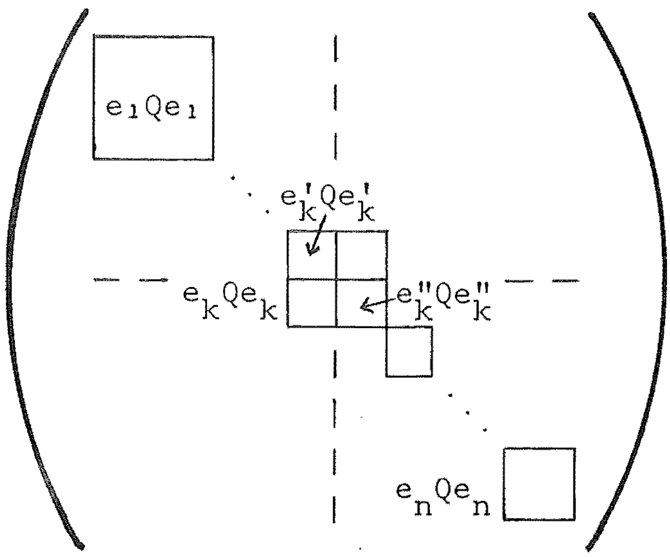
<u>Remark</u>. Notice that Rad $P = \sum_{i<j} e_i Q e_j$ which is nilpotent of index n.

<u>Proof</u>. Suppose P is a right order in Q. Since $e_1 Q + Q(1 - e_1) \supseteq P$ and $(1 - e_n)Q + Qe_n \supseteq P$, it follows that $e_1 Q + Q(1 - e_1)$ and $(1 - e_n)Q + Qe_n$ are also right orders in Q. Therefore, by 2.1.2, $\dim e_1 Q > \dim (1 - e_1)Q$ and $\dim e_n Q \geq \aleph_0$. Moreover, since $\dim (1 - e_1)Q = \dim e_2 Q + \ldots + \dim e_n Q$, we have $\dim e_1 Q > \dim e_i Q$ for $i = 2, \ldots, n$.

To show the converse we shall proceed by induction on the number n of idempotents $e_1, \ldots, e_n$. Corollary 2.1.2 says the theorem holds for $n = 2$. Explicitly, our induction hypothesis is: whenever a left full linear ring L contains nonzero orthogonal idempotents $f_1, \ldots, f_m$, $2 \leq m < n$, with $f_1 + \ldots + f_m = 1$, $\dim f_1 L > \dim f_i L$ for $i = 2, \ldots, m$ and $\dim f_m L \geq \aleph_0$, then $\sum_{i \leq j} f_i L f_j$ is a right order in L. Let Q and $e_1, \ldots, e_n$ be given as in the statement of the theorem. We consider two cases.

Case 1. Suppose $\dim e_iQ \leqslant \dim e_nQ$ for $i = 2,\ldots,n$. In this case no induction is required. For $\dim e_nQ = \dim (1-e_1)Q$ and hence, by theorem 2.1.1, $e_1Q + Qe_n$ is a right order in Q. Since $P \supseteq e_1Q + Qe_n$, P is also a right order in Q.

Case 2. Suppose there exists an integer k, $1 < k < n$, such that $\dim e_kQ > \dim e_nQ$. Choose k to be the largest such integer. We can now make the induction step by splitting up the "k^{th} diagonal block" e_kQe_k and applying lemma 2.1.3. The diagram below may be helpful.



First observe that for any nonzero idempotent e of Q, eQe is a left full linear ring and $\dim (eQ)_Q = \dim (eQe)_{eQe}$. This can be seen by noting that $(Q \nabla Qe)_{Qe}$ and hence, by the isomorphism which exists between the lattice of closed right ideals of Q and that of Qe (proposition 1.1.3), $\dim (eQ)_Q = \dim (eQ \cap Qe)_{Qe}$, that is, $\dim (eQ)_Q = \dim (eQe)_{eQe}$. Let $\aleph = \dim e_kQ$. Since $\aleph$ is infinite, we can choose orthogonal idempotents $e_k', e_k'' \in e_kQe_k$ such that $\dim e_k'Q = \dim e_k''Q = \aleph$ and $e_k = e_k' + e_k''$.

Let $e = e_1 + e_2 + \ldots + e_{k-1} + e_k'$. Then

$$1 - e = e_k'' + e_{k+1} + \ldots + e_n.$$

One can check that $e_1, e_2, \ldots, e_{k-1}, e_k'$, as orthogonal idempotents of eQe, satisfy the conditions of the induction hypothesis as applied to the ring eQe (in the order indicated), and likewise that $e_k'', e_{k+1}, \ldots, e_n$ (in this order) satisfy the conditions of the induction hypothesis as applied to the ring $(1-e)Q(1-e)$.

Applying the induction hypothesis to eQe and $(1-e)Q(1-e)$ respectively, yields:

$$T = \sum_{1 \leqslant i \leqslant j \leqslant k-1} e_iQe_j + \sum_{i=1}^{k-1} e_iQe_k' + e_k'Qe_k'$$

is a right order in eQe, and

$$K = \sum_{k+1 \leqslant i \leqslant j \leqslant n} e_i Q e_j + \sum_{j=k+1}^{n} e_k'' Q e_j + e_k'' Q e_k''$$

is a right order in $(1-e)Q(1-e)$.

Now $\dim eQ > \dim (1-e)Q \geqslant \aleph_0$ and clearly $_T(eQe \,\nabla\, T)$.

Hence applying lemma 2.1.3 we obtain that $T + eQ(1-e) + K$ is a right order in Q. Since $P \supseteq T + eQ(1-e) + K$, P is also a right order in Q. This completes the induction step and the proof of the theorem.

SECTION 2

Having exhibited right orders in Q which are not prime rings, let us see what we can say in general about such right orders.

LEMMA 2.2.1 Let R be a ring with $Z_r(R) = 0$ and let A be a right quotient ring of R. If A is a prime ring and if R contains no nonzero nilpotent ideals, then R is also a prime ring.

Proof. Suppose I and J are right ideals of R with $IJ = 0$. Let $K = JAI \cap R$. Then K is a nilpotent right ideal of R and therefore $K = 0$. Since $(A \,\nabla\, R)_R$, this implies $JAI = 0$. Hence, since A is a prime ring, we must have either $I = 0$ or $J = 0$. Thus R is a prime ring.

We now give a partial converse of theorem 2.1.4 (see remark following 2.1.4).

THEOREM 2.2.2 Let R be a right order in Q and suppose R contains a nilpotent ideal N of index $n > 1$. Then there exist nonzero orthogonal idempotents $e_1, \ldots, e_n$ of Q with $e_1 + \ldots + e_n = 1$, $\dim e_1 Q > \dim e_i Q$ for $i = 2, \ldots, n$, $\dim e_n Q \geqslant \aleph_0$ and such that

(i) $R \subseteq \sum_{i \leqslant j} e_i Q e_j$.

(ii) If $K = e_n Q e_n \cap R$ then $e_n Q e_n$ is a right quotient ring of K; in fact for each $x \in e_n Q e_n$ there exists $k \in K$ with k right invertible in $e_n Q e_n$ and $xk \in K$.

(iii) If N is a maximum nilpotent ideal of R then K is a prime ring.

Proof. Let s denote closure in the lattice $L(Q_R)$. Recall that for a right ideal I of R, I^s is actually a closed right ideal of Q (proposition 1.1.3), and hence I^s is a principal right ideal of Q (proposition 1.1.4). We define closed right ideals $A_1, \ldots, A_{n-1}$ of Q inductively as follows. Let $A_1 = (N^{n-1})^s$. A_2 is then chosen such that $(N^{n-2})^s = A_1 \dotplus A_2$, or in general, having chosen $A_1, \ldots, A_{i-1}$ such that
$(N^{n-(i-1)})^s = A_1 \dotplus A_2 \dotplus \ldots A_{i-1}$ we choose A_i such that $(N^{n-i})^s = A_1 \dotplus A_2 \dotplus \ldots \dotplus A_i$.

Since $N^i N^{n-i} = 0$ but $N^i N^{n-(i-1)} \neq 0$, we have

$(N^{n-i})^s \neq (N^{n-(i-1)})^s$ for $i = 1,\dots,n-1$ (see proposition 1.1.2). Thus $A_i \neq 0$ for $i = 1,\dots,n-1$. Choose orthogonal idempotents $e_1,\dots,e_n$ of Q such that $e_iQ = A_i$ for $i = 1,\dots,n-1$ and $e_1 + \dots + e_n = 1$. Since $(N^{n-i})^s = e_1Q \dotplus \dots \dotplus e_iQ$, $(e_{i+1}Q \dotplus \dots \dotplus e_nQ) \cap R \dotplus N^{n-i}$ is a large right ideal of R. Also,

$$(e_{i+1} + \dots + e_n)R(e_1 + \dots + e_i)\{(e_{i+1}Q + \dots + e_nQ) \cap R + N^{n-i}\} = 0$$

because N^{n-i} is a two-sided ideal of R. But $Z(Q_R) = 0$ and hence $(e_{i+1} + \dots + e_n)R(e_1 + \dots + e_i) = 0$ for for $i = 1,\dots,n-1$. Thus for $i > j$, $e_iRe_j = e_i(e_{j+1} + \dots + e_n)R(e_1 + \dots e_j)e_j = 0$. Hence $R \subseteq \sum_{i \leqslant j} e_iQe_j$ and consequently $\sum_{i \leqslant j} e_iQe_j$ is also a right order in Q. By theorem 2.1.4, we conclude that $\dim e_1Q > \dim e_iQ$ for $i = 2,\dots,n$ and $\dim e_nQ \geqslant \aleph_0$. This proves (i).

Let $A = \{r \in R : e_nr \in R\}$. A is a right ideal of R and contains a regular element of R. Hence A is also a right order in Q. Let $x \in e_nQe_n$ be given. Then there exists a regular element c of A such that $xc \in A$. Let $k = e_nc = e_nce_n$. Then $k \in K$ and $xk = xe_nc = xc \in K$. Furthermore, $\ell(k,Q) \cap e_nQe_n = 0$ because $\ell(c,Q) = 0$. Thus k has a right inverse in e_nQe_n. This proves (ii).

Suppose N is a maximum nilpotent ideal of R. Our construction shows that $N \subseteq (1 - e_n)Q(1 - e_1)$. Now

$K = e_n Q \cap R$ so that K is a right ideal of R. Hence if I is a nilpotent ideal of K then IK is a nilpotent right ideal of R. But by assumption, N contains all the nilpotent right ideals of R. Hence $IK \subseteq N$, which implies $IK = (1-e_n)IK = 0$, that is, $IK = 0$. However, from (ii) we know that $Z_r(K) = 0$ and therefore $I = 0$. By lemma 2.2.1, K is a prime ring. This completes the proof of the theorem.

Remark. Let $T = e_1 R e_1$. Since $Re_1 = e_1 R e_1$, T is a subring of $e_1 Q e_1$. But even when N is a maximum nilpotent ideal, the ring $\sum_{i \leqslant j} e_i Q e_j$ given by our construction does not hug R closely enough for us to say much about T. Ideally, of course, one would like to be able to say that T is a prime ring and is a right order in $e_1 Q e_1$. It can be shown that if $T \subseteq R$ then T contains no nilpotent ideals. However, simple examples (similar to example 1 below) show that T need not be a right order in $e_1 Q e_1$.

There are two interesting corollaries to theorem 2.2.2.

COROLLARY 2.2.3 All right orders in Q are prime rings if and only if Q_Q is of finite or of countably infinite dimension.

Proof. If R is a right order in Q and R is not a prime ring, then R contains nonzero nilpotent ideals by lemma 2.2.1. Theorem 2.2.2 now tells us that dim Q must be uncountable. The converse follows immediately from theorem 2.1.1.

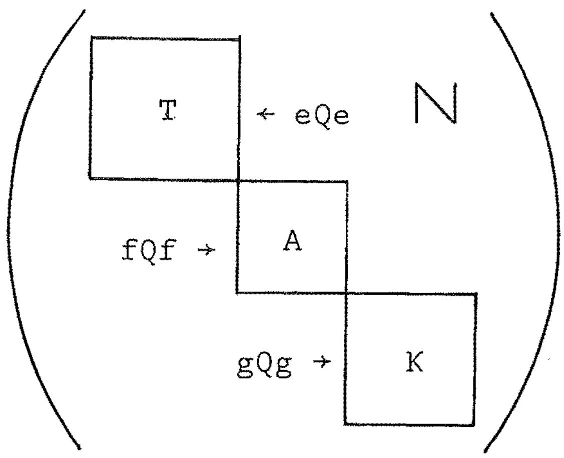
COROLLARY 2.2.4. If R is both a left and right order in Q, then R is a prime ring.

Proof. Let us suppose that R is not a prime ring. Then by lemma 2.2.1, R must contain a nonzero ideal N with $N^2 = 0$. Hence by theorem 2.2.2, there is an idempotent e of Q such that $R \subseteq eQ + Q(1-e)$, $\dim eQ = \dim Q$ and $1 - e \neq 0$. Let $P = eQ + Q(1-e)$. Then P is also a left order in Q. By proposition 1.2.4, $QeQ = Q$. Hence there exist $x_1, \ldots, x_n \in Q$ and $a_1, \ldots, a_n \in eQ$ such that $1 = x_1a_1 + \ldots + x_na_n$. Choose a regular element $c \in P$ such that $cx_i \in P$ for $i = 1, \ldots, n$. Then $c \in PeQ$ and thus $PeQ = Q$. But this implies that $P \supseteq PeQ = Q$, that is, $P = Q$. Clearly this is impossible since $(1-e)Qe \neq 0$. Hence R is a prime ring.

If R is a non-prime right order in Q, then theorem 2.2.2 tells us roughly how R sits in Q, namely, as a subring of a block triangular matrix ring with the size of the blocks as described in the theorem. But apart from this, we have not said anything really significant about R. It would appear that there is a certain amount of pathology present, which makes a complete description of a right order in Q rather difficult if dim Q is uncountable. The next proposition illustrates the difficulties.

PROPOSITION 2.2.5 Suppose dim Q = $\aleph > \aleph_0$. Let $\aleph'$ and $\aleph''$ be any cardinals satisfying $\aleph > \aleph'' \geqslant \aleph' \geqslant \aleph_0$. Let e, f and g be orthogonal idempotents of Q with $e + f + g = 1$, dim eQ = $\aleph$, dim fQ = $\aleph'$ and dim gQ = $\aleph''$. Let $N = eQ(1 - e) + fQg$. If A is any subring of fQf and T and K are right orders in eQe and gQg respectively, then the ring $R = T + A + K + N$ is a right order in Q.

Remark. A matrix representation of R would look something like the diagram below.



<u>Proof</u>. Let $x \in Q$ be given. By theorem 2.1.1, there exists $a \in eQ + Qg$ such that a is a unit in Q and $xa \in eQ + Qg$. Let $xa = b$. Then, by using essentially the same argument as in the proof of lemma 2.1.3, but with three idempotents instead, there exists a unit $c \in Q$ such that ac, $bc \in R$. Then $x(ac) \in R$ and ac is a regular element of R. The only snag is showing that regular elements of R are units in Q. We shall show in Chapter III (proposition 3.3.4) that if $\dim Q$ is infinite, then any nonzero left ideal of Q intersects nontrivially with any right order in Q (only proposition 1.2.2 is required for the proof). Now let b be a regular element of R. We have shown that $(Q \nabla R)_R$. Hence $r(b,Q) = 0$. Let $I = \ell(b,Q)$ and suppose $I \neq 0$. Then $I \cap eQ \neq 0$. Pick $0 \neq a \in I \cap eQ$. Since $a \notin R$, $ae \neq 0$. By the result quoted above, there exists $\alpha \in eQe$ such that $\alpha ae \neq 0$ and $\alpha ae \in T$. But now $0 \neq \alpha a \in R$ and $\alpha ab = 0$. This is the contradiction we were seeking. Hence $\ell(b,Q) = 0$ and b is therefore a unit in Q. The proof is complete.

<u>Remarks</u>. (1) Note that $Q \cong eQe$ and that gQg (respectively fQf) is a left full linear ring with right dimension $\aleph''$ (respectively $\aleph'$). Hence a study of right orders in full linear rings of dimension $\aleph$ requires knowledge of right orders in full linear rings of dimension $\aleph''$.

(2) If dim Q is uncountable, then proposition 2.2.5 provides examples of right orders in Q which do not contain a maximum nilpotent ideal and whose Jacobson radical is not nil. For example, take $\aleph' = \aleph'' = \aleph_0$, $T = eQe$, $K = gQg$ and take A to be a radical ring, but not a nil ring, containing no maximum nilpotent ideal (for example, strictly upper triangular $\aleph_0 \times \aleph_0$ matrices). In general, for R as in 2.2.5, Rad R = Rad T + Rad A + Rad K + N.

If in theorem 2.2.2 it is assumed that N is a maximum nilpotent ideal of R, one might expect K to be a right order in e_nQe_n. We show, by a simple example, that this is not necessarily the case.

EXAMPLE 1. Let Q have uncountable dimension and let e, f and g be orthogonal idempotents of Q with $e + f + g = 1$, $\dim fQ = \dim gQ = \aleph_0$. Let $R = eQ + Qg +$ socle $(1-e)Q(1-e)$. Let $N = eQ(1-e)$. Then N is a maximum nilpotent ideal of R with $N^2 = 0$, but the construction given in the proof of theorem 2.2.2 could have given R as a subring of $\sum_{1 \leqslant i \leqslant j \leqslant 2} e_iQe_j$ where $e_1 = e$, $e_2 = 1 - e$, in which case $K = (e_2Qe_2)g +$ socle (e_2Qe_2) is not a right order in e_2Qe_2.

EXAMPLE 2. Let Q, e, f and g be the same as in example 1, and let $R = eQ + Qg$. In this case $N = eQ(1-e) + fQg$

is a maximum nilpotent ideal of R with $N^3 = 0$, $N^2 \neq 0$. Our construction could have recovered the idempotents e, f and g, that is, $e_1 = e$, $e_2 = f$, $e_3 = g$. If this were the case then $e_2Re_2 = 0$. Hence we can not hope to say much about the intermediate diagonal blocks e_iRe_i, $i = 2,\ldots,n-1$, arising from the idempotents $e_1,\ldots,e_n$ in theorem 2.2.2.

Let M be the maximum ideal of Q, that is, $M = \{x \in Q : \dim xQ < \dim Q\}$ if dim Q is infinite, and $M = 0$ if dim Q is finite. Let $\bar{Q} = Q/M$ and for a subset X of Q let $\bar{X}$ denote the image of X under the canonical mapping of Q onto $\bar{Q}$. With all the right orders R in Q that we have met so far $\bar{R} = \bar{Q}$, so that we have no evidence of misbehaviour in $\bar{R}$. The standard proof given to show that a right order in a simple Artinian ring must be a prime ring (see, for example, Goldie [1] theorem 13), actually shows that a right order A in any simple ring B with identity is a prime ring, even without the requirement that regular elements of A be units in B. We record a particular case of this observation in

<u>PROPOSITION 2.2.6</u> Let R be a right order in Q. Let M be the maximum ideal of Q and let $\bar{Q} = Q/M$. If $\bar{R}$ denotes the image of R under the canonical mapping of Q onto $\bar{Q}$,

then $\bar{R}$ is a prime ring.

Suppose R is a right order in Q and suppose R contains a nonzero nilpotent ideal. By theorem 2.2.2, there is an idempotent e of Q, $e \neq 0, 1$, such that $R \subseteq eQ + Q(1-e)$. Let $B = eQ \cap R$. B is an ideal of R and is closed as a right ideal of R (see proposition 1.1.3). Since $e \neq 0, 1$, $eQ + Q(1-e)$ is a proper subring of Q. Hence there exists a regular element $c \in R$ such that $c^{-1} \notin eQ + Q(1-e)$. Then $ce = ece$ but $c^{-1}e \neq ec^{-1}e$. Thus

$$eQ \supset ceQ \supset \ldots \supset c^n eQ \supset \ldots$$

is a strictly descending chain. By the lattice isomorphism of proposition 1.1.3,

$$eQ \cap R \supset (ceQ) \cap R \supset \ldots \supset (c^n eQ) \cap R \supset \ldots$$

is a strictly descending chain of closed right ideals of R. Now if s denotes closure in $L(R_R)$, then $(c^n B)^s = (c^n eQ) \cap R$. Thus we have

<u>PROPOSITION 2.2.7</u> If R is a right order in Q and if R is not a prime ring, then there exist a two-sided ideal B of R, B closed as a right ideal of R, and a regular element c of R such that

$$B \supset (cB)^s \supset \ldots \supset (c^n B)^s \supset \ldots$$

is a strictly descending chain of closed right ideals of R.

Of course, if R is a prime ring then the only non-zero ideal of R which is closed as a right ideal of R is R itself. Thus in a prime ring, the above chain does not get past the first term. It also follows from the above proposition that a right order in a simple Artinian ring is a prime ring since such a ring has minimum condition on closed right ideals (proposition 1.1.3).

CHAPTER III

INTRINSIC EXTENSIONS OF PRIME RINGS

In this chapter we study the nature of the requirement that regular elements of a right order R in Q be units in Q. Whereas this requirement is redundant if dim Q is finite, we show quite simply, in §3, that when dim Q is infinite it is such that nonzero left ideals of Q must have nonzero intersection with R, that is, Q is left intrinsic over R. We anticipate this in §§1 and 2 by studying intrinsic extensions of prime rings. Our principal result in this direction (theorem 3.2.1) permits a number of corollaries, among which are a result of Hutchinson when specialized to prime rings (3.2.8), a one-sided version of a theorem of Utumi (3.2.10), and corollary 3.2.9 which enables us to show in §3 that a prime right order in Q must have Q as a left quotient ring if dim Q is infinite. However, because of its possible independent interest, a more direct proof of 3.2.9 is given in §3 by using a result of Faith. Finally, §4 contains some observations concerning the rings K_∞ + socle D_∞, where K is a right order in D, D a division ring, and where K_∞ (resp. D_∞) denotes the ring of all $\aleph_0 \times \aleph_0$ column-finite matrices over K (resp. D).

Although only §§3 and 4 are directly concerned with right orders in full linear rings, we will have occasion to refer to some results of §§1 and 2 in later chapters.

SECTION 1

Following Faith and Utumi [1] we call a ring extension S of a ring R *right* (resp. *left*) *intrinsic over* R in case $A \cap R \neq 0$ for each nonzero right (resp. left) ideal A of S. Faith and Utumi in [1] studied the problem of when S right intrinsic over R implies S is a right quotient ring of R. We shall also be concerned with this problem in the case where R is a prime ring. Notice that even in this case the above implication is not automatic. For example, take S to be a field and R a proper subfield of S.

In this section we establish a result of Utumi on when the MRQ ring of a ring R with $Z_r(R) = 0$ is left intrinsic over R.

LEMMA 3.1.1 Let S be a ring with $Z_\ell(S) = 0$ and suppose S is left intrinsic over R. Let e be an idempotent of S and let $B = Se \cap R$. Then $r(B,S) = (1-e)S$.

Proof. Clearly $(1-e)S \subseteq r(B,S)$. Let $a \in r(B,S)$ and let $Y = \ell(a,S)$. Suppose $Se \not\subseteq Y$. Then $Se \cap Y \neq Se$ and since $Se \cap Y$ is a closed left ideal of S (proposition 1.1.2) we can find a nonzero left ideal K of S such that $K \subseteq Se$ and $K \cap Y = 0$. But then $(K \cap R)a = 0$ and $K \cap R \neq 0$, which contradicts $K \cap Y = 0$. Thus $Se \subseteq Y$ and therefore $a \in (1-e)S$. Hence $r(B,S) = (1-e)S$.

THEOREM 3.1.2 (Utumi [4] theorem 2.2). Let R be a ring with $Z_r(R) = 0$. Then the closed right ideals of R are its right annihilator ideals if and only if the MRQ ring of R is left intrinsic over R.

Proof. Let S be the MRQ ring of R. If B is a closed right ideal of R then $B = eS \cap R$ for some idempotent e of S (propositions 1.1.3 and 1.1.4). Hence, if S is left intrinsic over R then, by lemma 3.1.1, $B = r(S(1-e) \cap R, R)$. In particular, B is a right annihilator ideal of R.

Conversely, suppose the closed right ideals of R are right annihilator ideals. If S is not left intrinsic over R then there exists an idempotent e of S, $e \neq 0$, such that $Se \cap R = 0$. Let $K = (1-e)S \cap R$. Then K is a closed right ideal of R with $\ell(K,R) = Se \cap R = 0$. Since K is a right annihilator ideal of R, we must have $K = R$.

But this is clearly impossible because $eS \cap R \neq 0$. Thus S is left intrinsic over R and this completes the proof of 3.1.2.

It is appropriate here to state (without proof) a theorem of Utumi for which we offer a one-sided version in §2 (corollary 3.2.10).

<u>THEOREM 3.1.3</u> (Utumi [4] theorem 3.3). Let R be a ring with $Z_\ell(R) = Z_r(R) = 0$. Then every one-sided quotient ring of R is a two-sided quotient ring of R if and only if the closed one-sided ideals of R are the corresponding one-sided annihilator ideals.

For the sake of completeness we also state the main result of Faith and Utumi [1], which gives sufficient conditions for an intrinsic extension to be a quotient extension.

<u>THEOREM 3.1.4</u> (Faith and Utumi [1] theorem 3.1). Let R be a ring with $Z_r(R) = 0$. Suppose the MRQ ring of R is left intrinsic over R, and does not contain any nonzero strongly regular ideals (as rings). Then any right intrinsic extension of R is a right quotient ring of R.

<u>Remarks</u>. (1) If R is a prime ring, but not an integral domain, then its MRQ ring does not contain any nonzero

strongly regular ideals (see Utumi [2] theorem 4).

(2) Faith and Utumi describe their hypothesis, that the MRQ ring of R be left intrinsic over R, as being rather dubious. We shall see that for a prime ring R containing uniform right ideals, but not an integral domain, it is equivalent to saying that the MRQ ring of R is a left quotient ring of R. This is not satisfied, for example, even by the ring of all $\aleph_0 \times \aleph_0$ column-finite matrices over the integers.

SECTION 2

In the case of a prime ring R which has $Z_r(R) = 0$ and contains uniform right ideals, one can say a great deal more about when a right intrinsic extension of R is a right quotient ring of R. Our main result states:

<u>THEOREM 3.2.1</u> Let R be a prime ring but not an integral domain. If S is a ring which is right intrinsic over R and if S has $Z_r(S) = 0$ and contains uniform right ideals, then S is a right quotient ring of R.

The proof requires several lemmas.

<u>LEMMA 3.2.2</u> Suppose Q is a simple Artinian ring but not a division ring. Then Q is generated by its idempotents.

Proof. Let e be a primitive idempotent of Q. Since $(1-e)Q \neq 0$, $(1-e)Q$ contains a right ideal which is isomorphic to eQ. Hence there exist $\beta \in eQ(1-e)$, $\delta \in (1-e)Qe$ such that $\beta\delta = e$. Let $x \in eQ$. Then $x = exe + x(1-e) = \beta(\delta x e) + x(1-e)$ $= [e+\beta][1-e][e+\delta x e] + [e+x(1-e)][1-e]$ with each bracketed term an idempotent.
Since Q is equal to the sum of its minimal right ideals, it is clear that Q, as a ring, is generated by its idempotents.

LEMMA 3.2.3 Let S be a ring with identity, E a set of generators for S (as a ring) and R a subring of S containing units of S. Let $U = \{c \in R : c \text{ a unit in } S\}$ and $T = \{x \in S : xc \in R \text{ for some } c \in U\}$. If $E \subseteq T$ and if $c^{-1}Ec \subseteq E$ for all $c \in U$, then $S = T$.

Proof. Let $x \in T$ and $e \in E$. Then there exists $c \in U$ such that $xc \in R$. Choose $d \in U$ such that $(c^{-1}ec)d \in R$. Then $(xe)(cd) = (xc)(c^{-1}ec)d \in R$. Thus $xe \in T$. Hence finite products of elements in E belong to T. Let $e_1,\ldots,e_n \in E$ and $d \in U$. Then $(e_1e_2\ldots e_n)d = d(d^{-1}e_1d)\ldots(d^{-1}e_nd) \in T$. Now let $e_1,\ldots,e_n$ and $f_1,\ldots,f_m \in E$. Choose $d \in U$ such that $(f_1f_2\ldots f_m)d \in R$ and choose $c \in U$ such that $(e_1\ldots e_nd)c \in R$. Then $(e_1e_2\ldots e_n + f_1f_2\ldots f_m)dc \in R$

and thus $e_1e_2\ldots e_n + f_1f_2\ldots f_m \in T$. Since E is a set of generators for S, the result is now immediate.

<u>LEMMA 3.2.4</u> Let Q be a simple Artinian ring, but not a division ring, and let R be a subring of Q. If R is a prime ring and if Q is right intrinsic over R, then R is a right order in Q.

<u>Proof</u>. Let dim Q = n. Let e be a primitive idempotent of Q and let $U = eQ \cap R$. Observe that for $a \in U$, $a \neq 0$, Qa is a minimal left ideal of Q since aQ is a minimal right ideal of Q. Suppose we have found $a_1,\ldots,a_m \in U$, $m < n$, such that $Qa_1 + \ldots + Qa_m$ is a direct sum. Choose an idempotent f of Q such that $Qa_1 + \ldots + Qa_m = Qf$. Since Q is right intrinsic over R and R is a prime ring, we have $U[(1-f)Q \cap R] \neq 0$. Choose $a_{m+1} \in U$ such that $a_{m+1}(1-f) \neq 0$. Then $Qa_{m+1} \not\subseteq Qf$ and hence $Qa_{m+1} \cap Qf = 0$. Thus $Qa_1 + \ldots + Qa_m + Qa_{m+1}$ is a direct sum. Clearly then, there exist $a_1,\ldots,a_n \in U$ such that $Qa_1 \dotplus \ldots \dotplus Qa_n = Q$, and hence there exist orthogonal idempotents $f_1,\ldots,f_n$ of Q with $f_1 + \ldots + f_n = 1$ and $Qf_i \cap R \neq 0$ for $i = 1,\ldots,n$.

Now let g be an arbitrary nonzero idempotent of Q. Then there exist an integer k, $k \leqslant n$, and orthogonal primitive idempotents $g_1,\ldots,g_n$ of Q, such that $g = g_1 + \ldots + g_k$ and $1 = g_1 + \ldots + g_n$. Since $g_iQ \cap R \neq 0$

and $Qf_i \cap R \neq 0$, primeness of R implies $g_iQf_i \cap R \neq 0$ for $i = 1,\ldots,n$. Choose $c_i \in g_iQf_i \cap R$, $c_i \neq 0$, for $i = 1,\ldots,n$ and let $c = c_1 + \ldots + c_n$. Then c is a unit of Q and $gc = c_1 + \ldots + c_k \in R$. It now follows from lemmas 3.2.2 and 3.2.3 that R is a right order in Q.

<u>LEMMA 3.2.5</u> Suppose R is a prime ring and Q is a right quotient ring of R. If f is a primitive idempotent of Q such that $Qf \cap R \neq 0$, then $fQf \cap R$ is a right order in fQf.

<u>Proof</u>. Let $x \in fQf$, $x \neq 0$. Choose $r \in R$ such that $fr \in R$ and $0 \neq xr \in R$. Since R is a prime ring, $xr(Qf \cap R) \neq 0$. Hence there exists $y \in fQf \cap R$ such that $0 \neq xy \in fQf \cap R$. Since fQf is a division ring, it is clear that $fQf \cap R$ is a right order in fQf.

<u>LEMMA 3.2.6</u> Let R be a subring of Q and suppose Q is right intrinsic over R. Let e be a nonzero idempotent of Q and let $K = eQe \cap R$. If R is a prime ring and if $Qe = Qa$ for some $a \in R$, then K is a prime ring and eQe is right intrinsic over K.

<u>Proof</u>. Let $x \in eQe$, $x \neq 0$. Since $xQ \cap R \neq 0$ and $Qe \cap R \neq 0$, primeness of R implies that $(xQ \cap R)(Qe \cap R) \neq 0$. Hence $x(eQe) \cap K \neq 0$. Thus eQe is right intrinsic over

K. It follows that $Z_r(K) = 0$. Now suppose I and J are two-sided ideals of K with $IJ = 0$. Let $L = Qe \cap R$ and $B = eQ \cap R$. Then $(IB)(LJ) = 0$, which implies $IB = 0$ or $LJ = 0$ because R is a prime ring. $IB = 0$ implies $IK = 0$, which in turn implies $I = 0$ because $Z_r(K) = 0$. $LJ = 0$ implies $aJ = 0$, which implies $J = (1-e)J = 0$. Thus K is a prime ring and the proof is complete.

LEMMA 3.2.7 Let Q be a left full linear ring but not a division ring. If Q is right intrinsic over a subring R and if R is a prime ring, then Q is a right quotient ring of R.

Proof. Let V be a minimal right ideal of Q and let $U = V \cap R$. Choose $x \in U$, $x \neq 0$. By the argument used in the proof of lemma 3.2.4, we can find $y \in U$, $y \neq 0$, such that $Qx \cap Qy = 0$. Choose orthogonal idempotents f and g of Q such that $Qx = Qf$ and $Qy = Qg$. Then $Qf \cap R \neq 0$ and $Qg \cap R \neq 0$. Hence, since Q is right intrinsic over R and R is a prime ring, we have $fQf \cap R \neq 0$ and $gQg \cap R \neq 0$. Let $e = f + g$. Then it is clear that there exists $a \in R$ such that $Qa = Qe$. Let $K = eQe \cap R$. By lemma 3.2.6, K is a prime ring and eQe is right intrinsic over K. Hence, since eQe is a simple Artinian ring but not a division ring, K is a right order in eQe by lemma 3.2.4.

By lemma 3.2.5, $fKf \cap K$ is a right order in $f(eQe)f = fQf$. In particular, $fQf \cap R$ is a right order in fQf.

Now let $x \in Qf, x \neq 0$. Since $(xQ \cap R)(Qf \cap R) \neq 0$, there exists $b \in fQf$ such that $0 \neq xb \in Qf \cap R$. $fQf \cap R$ being a right order in fQf, we can choose $y \in fQf \cap R$ such that $0 \neq by \in fQf \cap R$. Since Qf is a right vector space over fQf, we have $x(by) \neq 0$ and thus $0 \neq x(by) \in x(Qf \cap R) \cap (Qf \cap R)$. Hence Qf is a right quotient ring of $Qf \cap R$. Since Q is a prime ring, Q is a right quotient ring of Qf and it follows that Q is therefore a right quotient ring of R. This completes the proof of 3.2.7.

We now have all the lemmas we need to prove theorem 3.2.1.

Proof of Theorem 3.2.1. Let T be the MRQ ring of S. It is easily seen that T is a prime ring but not an integral domain. By propositions 1.1.3 and 1.1.4, T has nonzero socle. Thus T is a left full linear ring by proposition 1.2.2. Clearly T is also right intrinsic over R and hence, by lemma 3.2.7, T is a right quotient ring of R. In particular, S is a right quotient ring of R and this completes the proof of 3.2.1.

Hutchinson in [1] characterized intrinsic extensions of semiprime right Goldie rings. Our first corollary to theorem 3.2.1 is Hutchinson's result when specialized to prime right Goldie rings.

<u>COROLLARY 3.2.8</u> (Hutchinson [1] theorem 4.5). Let R be a prime right Goldie ring but not an integral domain. If S is a ring which is right intrinsic over R and if $Z_r(S) = 0$, then S is a right quotient ring of R.

<u>Proof</u>. All we need to show is that S contains uniform right ideals. However, since an infinite direct sum of nonzero right ideals of S would give rise to an infinite direct sum of nonzero right ideals of R, it is clear that S has finite right dimension. Thus, by a standard argument, S contains uniform right ideals.

Our next corollary is an immediate consequence of theorem 3.2.1.

<u>COROLLARY 3.2.9</u> Let R be a prime ring with $Z_r(R) = 0$, but not an integral domain, and suppose R contains uniform right ideals. Let S be the MRQ ring of R. Then S is a left quotient ring of R if and only if S is left intrinsic over R.

We restate corollary 3.2.9, using theorem 3.1.2, in order to view it as a one-sided version of Utumi's result, theorem 3.1.3.

COROLLARY 3.2.10 Let R be a prime ring with $Z_r(R) = 0$, but not an integral domain, and suppose R contains uniform right ideals. Then every right quotient ring of R is also a left quotient ring of R if and only if the closed right ideals of R are its right annihilator ideals.

Our final corollary concerns when a prime right Goldie ring is also a left Goldie ring. The result is probably well known, although in Goldie [2] the problem of when a right order R in a simple Artinian ring is also a left order (theorem 5.6) is considered separately from the problem of when closed (= complement) right ideals of R are right annihilator ideals (theorem 3.12).

COROLLARY 3.2.11 Let R be a prime right Goldie ring but not an integral domain. Then R is a left Goldie ring if and only if the closed right ideals of R are the right annihilator ideals.

Proof. By theorem 3.1.2, if R is a left Goldie ring then the closed right ideals of R are its right annihilator ideals. The converse follows from corollary 3.2.10 (actually, only theorem 3.1.2 and lemma 3.2.4 are needed).

An obvious example of where the one-sided version of Utumi's result (3.1.3) fails, is obtained by choosing R to be a right Ore domain but not a left Ore domain. In view of the proof of theorem 3.2.1, one could possibly attribute the failure in this case to the fact that the MRQ ring of R is not generated by its idempotents. The following is an example of an irreducible ring R (see chapter I for definition), with dim $R_R = 2$, for which the one-sided version of 3.1.3 fails.

<u>EXAMPLE 1</u> Let K be a right Ore domain but not a left Ore domain, and let D be the right quotient division ring of K. Let $S = D_2$ (the ring of all 2×2 matrices over D) and let R be the subring of S which consists of all matrices of the form

$$\begin{pmatrix} k & a \\ o & b \end{pmatrix}$$

where $k \in K$ and $a, b \in D$. Since R contains a nonzero left ideal of S, S is a right quotient ring of R. Thus S is the MRQ ring of R and dim $R_R = 2$. However, under the natural embedding of K in R left ideals of K remain left ideals of R, so that dim ${}_R R$ is infinite. So S is certainly not a left quotient ring of R. But it is easily checked that S is left intrinsic over R. By theorem 3.1.2, the closed right ideals of R are its right annihilator ideals. Hence the one-sided version of 3.1.3 breaks down here.

SECTION 3

We begin this section with a result of Faith, which we will use to give a more direct proof of corollary 3.2.9.

THEOREM 3.3.1 (Faith [1] p.103). Let R be a prime ring with $Z_r(R) = 0$, and containing uniform right ideals. Let S be the MRQ ring of R and let e be a primitive idempotent of S. Let $U = eS \cap R$ and $D = eSe$. Then S is a left quotient ring of R if and only if

(i) Ue is a left Ore domain, and

(ii) $DU = eS$.

Proof. Suppose S is a left quotient ring of R. Let $d \in D$, $d \neq 0$. Choose $r \in R$ such that er, $dr \in R$ with $dr \neq 0$. Since ${}_R(S \nabla R)$ and R is a prime ring, we have $dr(Se \cap R) \neq 0$. Thus, letting $K = eSe \cap R$, we have $dK \cap K \neq 0$.

Similarly one shows that $Kd \cap K \neq 0$. Since D is a division ring and $K \subseteq Ue \subseteq D$, it is clear that Ue is both a left and a right Ore domain and D is its quotient division ring. Now let $x \in eS$, $x \neq 0$. Let $L = Sx \cap R$. Then $L \neq 0$ and by primeness of R, $UL \neq 0$. Choose $u \in UL$, $u \neq 0$. Then $u \in Dx$ and since ${}_D(eS)$ is a vector space, this implies $x \in Du$. Thus $eS \subseteq DU$ and hence $eS = DU$.

Conversely, let us suppose that (i) and (ii) are satisfied. Since $(S \nabla R)_R$ and R is a prime ring, a calculation similar to the one above shows that Ue is a right Ore domain with right quotient division ring D. Hence (i) implies that D is also the left quotient division ring of Ue.

Now let $y \in S$, $y \neq 0$. Then $(S \nabla R)_R$ and R prime imply $Uy \neq 0$. Choose $u \in U$ such that $uy \neq 0$. By (ii) $uy = d_1u_1 + \ldots + d_nu_n$ for some $d_i \in D$, $u_i \in U$, $i = 1,\ldots,n$. Since Ue is a left order in D, there exists $\bar{u} \in U$, such that $\bar{u}e \neq 0$ and $(\bar{u}e)d_i \in Ue$ for $i = 1,\ldots,n$. Then $0 \neq \bar{u}uy \in Ry \cap R$. It is now clear that S is a left quotient ring of R. This completes the proof of 3.3.1.

<u>LEMMA 3.3.2</u> For R, S, e and U as in 3.3.1, S is a left quotient ring of R if and only if S is left intrinsic over R and Ue is a left Ore domain.

Proof. The "only if" part follows immediately from 3.3.1. Conversely, if S is left intrinsic over R and Ue is a left Ore domain, then one can check that the proof used in showing the necessity of (ii) in theorem 3.3.1 goes over without change. The result now follows from that theorem.

3.3.3 Direct proof of 3.2.9 By propositions 1.1.3 and 1.2.2, S is a left full linear ring but not a division ring. Let e be a primitive idempotent of S and let $U = eS \cap R$. Let $D = eSe$.

Let us suppose that S is left intrinsic over R but not a left quotient ring of R. Then by lemma 3.3.2, Ue is not a left Ore domain. Hence there exist nonzero $k_1, k_2 \in Ue$ such that $Uek_1 \cap Uek_2 = 0$, that is, $Uk_1 \cap Uk_2 = 0$. Since S is left intrinsic over R and R is a prime ring, we have $(1-e)Se \cap R \neq 0$ ($1-e \neq 0$ because S is not a division ring). Choose $\beta \in (1-e)Se \cap R$, $\beta \neq 0$, and choose orthogonal idempotents $f, g \in S$ such that $fS = \beta S$ and $e + f + g = 1$ (g is possibly 0). Choose $\delta \in eSf$ such that $\delta\beta = e$, that is, δ provides the inverse of the isomorphism $x \to \beta x$ of eS onto fS. Let k_1^{-1}, k_2^{-1}, denote the inverses of k_1, k_2 in D. Since $k_1^{-1} - k_2^{-1}\delta \neq 0$ and S is left intrinsic over R, we have $S(k_1^{-1} - k_2^{-1}\delta) \cap R \neq 0$. Choose $x \in S$ such that $0 \neq x(k_1^{-1} - k_2^{-1}\delta) \in R$ and let $a = x(k_1^{-1} - k_2^{-1}\delta)$.

Then $ak_1 = xe$ and $a\beta k_2 = -xe$. Thus $ak_1 + a\beta k_2 = 0$. Furthermore $ag = 0$ so that $a = ae + af \in$ socle S.

Choose nonzero orthogonal primitive idempotents $e_1, \ldots, e_n \in S$ such that $a \in e_1 S + \ldots + e_n S$. S left intrinsic over R and R a prime ring imply $eSe_i \cap R \neq 0$ for $i = 1, \ldots, n$. Choose $\delta_i \in eSe_i \cap R$ and $\beta_i \in e_i Se$, $i = 1, \ldots, n$, such that $\beta_i \delta_i = e_i$. Since $ak_1 + a\beta k_2 = 0$, we have $\delta_i ak_1 + \delta_i a\beta k_2 = 0$. Moreover, $\delta_i a \in U$ and $\delta_i a\beta \in U$. Hence, since $Uk_1 \cap Uk_2 = 0$, we have $\delta_i ak_1 = \delta_i a\beta k_2 = 0$ for $i = 1, \ldots, n$. Hence $\delta_i ae = 0$ and $\delta_i a\beta = 0$. Since $\beta S = fS$, $\delta_i a\beta = 0$ implies $\delta_i af = 0$. Hence $\beta_i \delta_i ae = \beta_i \delta_i af = 0$, which implies $e_i ae = e_i af = 0$ for $i = 1, \ldots, n$. Furthermore, $ag = 0$ implies $e_i a = e_i ae + e_i af = 0$ for $i = 1, \ldots, n$. Thus, $a = e_1 a + \ldots + e_n a = 0$. But a was chosen to be nonzero. From this contradiction we deduce that S is a left quotient ring of R if S is left intrinsic over R. The converse is obvious so the proof is complete.

<u>Remarks</u> (1) Cateforis and Sandomierski [1] have recently proved the following interesting result (theorem 1.1): let R be a ring with identity and $Z_r(R) = 0$, and let S be the MRQ ring of R. Then S is a left quotient ring of R if and only if every finitely generated right R-module with zero singular submodule is torsionless (that

is, can be embedded in a direct product of copies of R_R).

(2) See also Cateforis [3] theorem 2.3. There it is shown that if $Z_r(R) = 0$ and $\dim R_R < \infty$, then S (as in (1)) is a left quotient ring of R if and only if S_R is flat. However, an obvious example, by taking R to be a regular left self-injective ring which is not right self-injective, shows that the assumption that R_R be finite dimensional is, in general, essential for this result.

We now return to the study of right orders in a left full linear ring Q.

PROPOSITION 3.3.4 Suppose dim Q is infinite. If R is a right order in Q then Q is left intrinsic over R.

Proof. Let us suppose that Q is not left intrinsic over R. Since Q has nonzero socle, this means that we can find a primitive idempotent e of Q such that $Qe \cap R = 0$. By proposition 1.2.3 we must have $\dim (1-e)Q = \dim Q$ and hence $Q_Q \cong (1-e)Q$. Hence there exists an element $c \in Q$ with $c^r = 0$ and $cQ = (1-e)Q$, that is, $c^\ell = Qe$. Since R is a right order in Q, there exists $a \in R$ with a invertible in Q and $ca \in R$. Let $y = ca$. Then $\ell(y,R) = \ell(c,R) = Qe \cap R = 0$ and $r(y,R) = 0$, that is, y is a regular element of R. But since

$\ell(y,Q) = Qe \neq 0$, y is not regular in Q. Thus R has failed to meet one of the requirements for a right order in Q. We conclude, therefore, that Q is left intrinsic over R.

Combining 3.3.4 with 3.2.9 and 2.2.3, we obtain the main result of this section.

THEOREM 3.3.5 Let R be a right order in Q. If dim Q is countable then Q is a left quotient ring of R. In general, if dim Q is infinite and if R is a prime ring, then Q is a left quotient ring of R.

SECTION 4

If dim Q is finite and if R is a right order in Q, then Q is a left quotient ring of R only when R is also a left order in Q. In this section we show, among other things, that this is no longer the case when the dim Q is infinite.

For a ring A, A_∞ will denote the ring of all $\aleph_0 \times \aleph_0$ column-finite matrices over A.

Let D be a division ring. If $Q = D_n$, $n < \infty$, then essentially the only way to construct a right order R in Q is to take a right order K in D and take R to be a subring of Q containing K_n (Faith and Utumi [2]). If

$Q = D_\infty$ and $R = K_\infty$, then the elements of Q can still be expressed in the form ab^{-1}, a and b in R with b invertible in Q, but unless each countable collection of nonzero left ideals of K has nonzero intersection, there will be elements of K_∞ which are regular K_∞ but not in D_∞, or, in the language of §3, D_∞ will not be left intrinsic over K_∞. For suppose there exist nonzero elements k_i of K, $i = 1,\ldots,\infty$, such that $\bigcap_1^\infty Kk_i = 0$. Let e be the matrix in $Q(= D_\infty)$ whose first row is $(1, k_1^{-1}, k_2^{-1},\ldots, k_i^{-1}, \ldots)$ and whose other rows consist of all zeros. Then it is easily seen that $Qe \cap R = 0$, which contradicts proposition 3.3.4. Alternatively, the element y given by the matrix below

$$\begin{pmatrix} -1 & -1 & -1 & \cdots & -1 & \cdots \\ k_1 & 0 & 0 & & & \\ 0 & k_2 & 0 & & & \\ \vdots & & \ddots & & & \\ & & & k_i & & \\ & & & & \ddots & \end{pmatrix}$$

is regular in R but not in Q, since $\ell(y,Q) = Qe$, $r(y,Q) = 0$. The above observations form the first half of the following proposition. The proof of the second half is straight forward and will be omitted.

PROPOSITION 3.4.1 Let D be a division ring and let K be a right order in D. Let $Q = D_\infty$ and let $R = K_\infty$. If R is a right order in Q then each countable collection of nonzero left ideals of K has nonzero intersection. Conversely, if K has this property then R is a right order and also a left order in Q.

Suppose R is a subring of Q such that the elements of Q can be expressed in the form ab^{-1}, a and b in R with b invertible in Q. Even if R does not meet the requirement that its regular elements be units in Q, R+ socle Q certainly does. Hence R + socle Q is a right order in Q. Thus the first part of the next proposition is clear.

PROPOSITION 3.4.2 Let D be a division ring and let K be a right order in D. Let $Q = D_\infty$ and let $R = K_\infty +$ socle Q. Then R is a right order in Q. Furthermore, R is also a left order in Q if and only if K has the countable intersection property for left ideals.

Proof. Suppose R is a left order in Q and suppose K does not satisfy the countable intersection property.

Then there exists a set $\{k_i\}_{i \in N}$ of nonzero elements of K, where N is the set of natural numbers, such that $\bigcap_{i \in N} Kk_i = 0$. Write $N = \bigcup_{i \in N} N_i$ where each N_i is infinite and $N_i \cap N_j = \phi$ if $i \neq j$. For each $i \in N$, choose a 1-1 onto map $f(i): N_i \to N$. Define $x \in Q$ to be the matrix (x_{ij}) where $x_{ij} = k^{-1}_{f(i)(j)}$ if $j \in N_i$, otherwise $x_{ij} = 0$. Then it is easily seen that for $y \in R$, $yx \in R$ implies $y \in$ socle Q. This is the desired contradiction and hence the proof of 3.4.2 is complete.

Remarks. (1) There do exist integral domains (commutative in fact) with the countable intersection property for left ideals. See Faith [1], p.129, problem 12.

(2) Let D and K be as in 3.4.2. Let $\aleph$ be an infinite cardinal and let Q(resp. R) be the ring of all $\aleph \times \aleph$ column-finite matrices over D (resp. K). Then it is clear how the proof of 3.4.2 could be modified so as to show that $R +$ socle Q is a left order in Q if and only if for each collection $\{I_\alpha\}_{\alpha \in \Omega}$ of nonzero left ideals of K with $|\Omega| = \aleph$, $\bigcap_{\alpha \in \Omega} I_\alpha \neq 0$.

CHAPTER IV

FINITENESS CONDITIONS

When Q has finite dimension, right orders in Q are distinguishable from the other subrings of Q over which Q is a right quotient ring by their possession of a property, primeness, which is not a finiteness property. However, if dim Q is infinite and Q is a right quotient ring of R, then it seems unlikely that there is a reasonable non-finiteness condition which, when satisfied by R, will ensure that R is a right order in Q (after all, there are regular rings with identity which have Q as a proper right quotient ring). The relation "Q is a right quotient ring of R" is just too weak in this case to give any significant similarities in the right ideal structures of Q and R. Specifically, the map ϕ which sends a right ideal B of Q to $B \cap R$ need not be one-to-one, and the map Ψ which sends a right ideal I of R to IQ need not preserve intersections. We point out now, that requiring both of these properties for a ring R with identity is equivalent, in the language of Findlay [1], to requiring that Q be the (left) flat epimorphic hull of R.

The contents of the present chapter are briefly as follows. Suppose Q is a right quotient ring of R. In §1 we introduce a finiteness condition (A), which for R is equivalent to ϕ being one-to-one. In §2 we ask: if R is a prime ring which has Q as a two-sided quotient ring, and if R + socle Q is a right order in Q, is R already a right order in Q? We give an affirmative answer if R satisfies (A) or if R has a classical right quotient ring. In §3 we take a closer look at the condition (A) and also introduce the notion of a reducing pair of elements in a ring. When dim Q is infinite, this notion plays a role similar to that of primeness when dim Q is finite, in so far as determining whether R is a right order in Q. If R satisfies (A) and contains a reducing pair of elements, then Ψ preserves intersections. Actually, theorem 4.3.7 and corollary 4.3.8 say much more than this. Finally, in §4 we relate the condition (A) on R to Q being a flat epimorphic extension of R.

SECTION 1

Let M_R be a right module over a ring R. M_R is said to be *essentially finitely generated* if there exist $x_1,\ldots,x_n \in M$ such that $x_1R + \ldots + x_nR$ is an essential submodule of M_R. M_R is said to be *essentially principally generated* if there exists $x \in M$ such that xR is an essential submodule of M_R. We make the following definition.

DEFINITION 4.1.1 A ring R is said to satisfy (A) (resp. (A_1)) if $Z_r(R) = 0$ and each closed right ideal of R is essentially finitely generated (resp. essentially principally generated) as a right R-module.

PROPOSITION 4.1.2 (Cateforis [2] lemma 2.4). Suppose Q is a right quotient ring of R. Then R satisfies (A) if and only if $(B \cap R)Q = B$ for all right ideals B of Q, that is, if and only if the map $B \to B \cap R$ is one-to-one.

Proof. Suppose R satisfies (A). Let B be a right ideal of Q. Let $x \in B$. Then (A) implies that there exist $x_1, \ldots, x_n$ in $xQ \cap R$ such that $x_1R + \ldots + x_nR$ is an essential R-submodule of $xQ \cap R$. Then $x_1Q + \ldots + x_nQ = xQ$ so that $x \in (B \cap R)Q$. Thus $B \subseteq (B \cap R)Q$. Clearly, $(B \cap R)Q \subseteq B$. Hence $(B \cap R)Q = B$.

To establish the converse, let K be a closed right ideal of R. Then $K = eQ \cap R$ for some idempotent e of Q. Since $eQ = KQ$, there exist $x_1, \ldots, x_n \in K$ such that $eQ = x_1Q + \ldots + x_nQ$. It follows that $x_1R + \ldots + x_nR$ is an essential submodule of K_R. Hence R satisfies (A).

<u>PROPOSITION 4.1.3</u> Suppose Q is a right quotient ring of R and also left intrinsic over R. Then the following are equivalent.

(i) R satisfies (A_1).

(ii) The closed right ideals of R are of the form $a^{\ell r}$, $a \in R$, where annihilators are taken in R.

(iii) The left annihilator ideals of R are of the form a^{ℓ}, $a \in R$.

<u>Proof</u>. (i) implies (ii). Let K be a closed right ideal of R. Then there exists $a \in K$ such that aR is essential in K_R. By proposition 1.1.2, $a^{\ell} = K^{\ell}$. By theorem 3.1.2, $K = K^{\ell r}$ and hence $K = a^{\ell r}$. This gives (ii).

(ii) implies (iii). Let J be a left annihilator ideal of R. By (ii), there exists $a \in R$ such that $J^r = a^{\ell r}$. Hence $J = J^{r\ell} = a^{\ell r \ell} = a^{\ell}$. This given (iii).

(iii) implies (i). Let K be a closed right ideal of R. By theorem 3.1.2, $K = K^{\ell r}$. Now (iii) implies that there exists $a \in R$ such that $K^{\ell} = a^{\ell} = (aR)^{\ell}$. Since $a \in K^{\ell r}$, we have $a \in K$. By proposition 1.1.2 and theorem 3.1.2, aR is essential in $(aR)^{\ell r}$. Hence aR is essential in K_R. This gives (i) and completes the proof of 4.1.3.

PROPOSITION 4.1.4 If R is a right order in Q, then R satisfies (A_1). Hence if dim Q is infinite, R satisfies (ii) and (iii) of 4.1.3.

Proof. Let K be a closed right ideal of R. By propositions 1.1.3 and 1.1.4, there exists an idempotent e of Q such that $K = eQ \cap R$. Choose a regular element $c \in R$ such that $ec \in R$. Let $a = ec$. Then $a \in K$ and $aQ = eQ$. It follows that aR is essential in K_R. Thus R satisfies (A_1).

If dim Q is infinite, then Q is left intrinsic over R by proposition 3.3.4. Hence R satisfies (ii) and (iii) of 4.1.3.

Remarks (1) The notion of an essentially finitely generated module plays an important role in Cateforis [1], [2].

(2) If R has $Z_r(R) = 0$ and finite right dimension, then R satisfies (A). In fact, every right ideal of R is essentially finitely generated (see Sandomierski [1]). However, R need not satisfy (A_1) (see example 1 of §3).

(3) Johnson in [5] calls a ring which has finite right dimension and which satisfies (A_1), an I_r-ring. See his corollary to theorem 3, p.286.

(4) Condition (A) is not automatically satisfied by a ring R if dim R_R is infinite. For example, let Q have infinite dimension and let R = eQ + socle Q, where e is an idempotent of Q with $(1-e)Q \cong eQ$. Then $(1-e)R$ is a closed right ideal of R which is not essentially finitely generated.

SECTION 2

In this section, R will denote a prime ring which has Q as a left and right quotient ring. Chase and Faith [1] theorem 4.5, have given a nice description of such rings (see also Koh and Mewborn [1] theorem 2). Such a description, of course, says only what R looks like inside the socle of Q. The object of this section is to show that if R satisfies (A) then we can suppose that R actually contains the socle of Q (equivalently, socle R ≠ 0) if we wish to show that R is a right order in Q (theorem 4.2.8).

We begin with some lemmas which will be needed in proving a result about lifting units of Q/socle Q (theorem 4.2.4).

LEMMA 4.2.1 Let e be an idempotent of Q. If f is an idempotent of Q such that fQ = eQ then dim $(1-f)Q$ = dim $(1-e)Q$. If g is an idempotent of Q such that

$gQ + eQ = Q$ then $\dim gQ \geq \dim (1-e)Q$.

Proof. $fQ = eQ$ implies $(1-f)Q \cong (1-e)Q$. Hence $\dim (1-f)Q = \dim (1-e)Q$.

$gQ + eQ = Q$ implies $1-e = gx$ for some $x \in Q$. Since $(gx)^r \cap (1-e)Q = 0$, left multiplication by gx maps $(1-e)Q$ isomorphically into gQ. Thus $\dim gQ \geq \dim (1-e)Q$.

For an element $y \in Q$ we shall define codim (y) to be the dimension of any complement of yQ, that is, if e is an idempotent of Q such that $yQ = eQ$ then codim $(y) = \dim (1-e)Q$. This is well defined because of lemma 4.2.1. If codim (y) is finite then codim $(y) = \dim y^{\ell}$, but if codim (y) is infinite then this is not necessarily so. For example, codim $(0) = \dim Q \neq \dim {}_Q Q$ unless $\dim Q$ is finite.

LEMMA 4.2.2 Let $y \in Q$ and suppose $\dim y^r >$ codim (y). Then for any $x \in$ socle Q, $(x+y)^r \neq 0$.

Proof. Since Q is a regular ring, there exist idempotents $e, f \in Q$ such that $yQ = eQ$ and $Qy = Qf$. In terms of e and f, $\dim y^r >$ codim (y) means $\dim (1-f)Q > \dim (1-e)Q$. Let $x \in$ socle Q be given. Let $h = ex$. Since $y(fQ) = eQ$, there exists $a \in fQ$ such that $ya = e$. Let $h_1 = ah$. Then $h_1Q \subseteq fQ$ and $yh_1Q = hQ$.

Moreover, since left multiplication by y maps fQ isomorphically onto eQ, we have $\dim h_1Q = \dim hQ$.

We now choose idempotents $g, g_1 \in Q$ such that $gQ = (1-e)Q \dotplus hQ$ and $g_1Q = (1-f)Q \dotplus h_1Q$. Then we have $(x+y)g_1 \in gQ$. Since $\dim h_1Q = \dim hQ < \infty$, we have $\dim g_1Q = \dim (1-f)Q + \dim h_1Q > \dim (1-e)Q + \dim hQ = \dim gQ$, that is, $\dim g_1Q > \dim gQ$. Thus left multiplication by $(x+y)g_1$ cannot map g_1Q isomorphically into gQ. Hence there exists $z \in g_1Q$, $z \neq 0$, such that $(x+y)g_1z = 0$, that is, $(x+y)z \neq 0$. Thus $(x+y)^r \neq 0$ and this completes the proof of 4.2.2.

<u>Remark</u>. When $\dim Q$ is finite, $Q =$ socle Q and so $\dim y^r = \operatorname{codim}(y) = \dim y^\ell$ for all $y \in Q$. However when $\dim Q$ is infinite, the situation where $\dim y^r > \operatorname{codim}(y)$ (and vice versa) arises frequently. For example, let D be a division ring and let $Q = D_\infty$. Let y be the matrix

$$\begin{pmatrix} 0 & 0 & 0 & \cdot & \cdot & \cdot & & \\ 0 & 0 & 1 & & & & & \\ 0 & 0 & 0 & 1 & & & & \\ \cdot & & & & 1 & & & \\ \cdot & & & & & \cdot & & \\ \cdot & & & & & & \cdot & \\ \cdot & & & & & & & \cdot \\ & & & & & & & \end{pmatrix}$$

Then $\dim y^r = 2$ whereas $\operatorname{codim}(y) = 1$.

<u>LEMMA 4.2.3</u> Let $y \in Q$ and suppose $\dim y^r < \operatorname{codim}(y)$. Then for any $x \in \text{socle } Q$, $(x+y)^\ell \neq 0$.

<u>Proof</u>. Let $x \in \text{socle } Q$ be given. If $xQ + yQ \neq Q$ then clearly $(x+y)^\ell \neq 0$. Hence we can suppose $xQ + yQ = Q$. Since $\dim xQ < \infty$, we have $\operatorname{codim}(y) < \infty$ by lemma 4.2.1. Now the proof of lemma 4.2.2 did not invoke the injectivity of Q_Q, only the fact that Q is a prime regular ring with nonzero socle. We conclude, therefore, that lemma 4.2.3 holds because we can take as our hypothesis $\dim y^r < \dim y^\ell < \infty$.

<u>THEOREM 4.2.4</u> Let y be an element of Q. Then there exists $x \in \text{socle } Q$ such that $x + y$ is a unit in Q if and only if $\dim y^r = \operatorname{codim}(y) < \infty$.

<u>Proof</u>. Suppose $x + y$ is a unit in Q for some $x \in \text{socle } Q$. Then by lemmas 4.2.2 and 4.2.3, $\dim y^r = \operatorname{codim}(y)$. By lemma 4.2.1, $\operatorname{codim}(y) < \infty$.

Conversely, suppose $\dim y^r = \operatorname{codim}(y) < \infty$. Choose idempotents $e, f \in Q$ such $yQ = eQ$ and $Qy = Qf$. Then $\dim (1-f)Q = \dim (1-e)Q < \infty$. Hence $(1-f)Q \cong (1-e)Q$ and so there exists $x \in (1-e)Q(1-f)$ such that $xQ = (1-e)Q$ and $Qx = Q(1-f)$. Clearly, $x \in \text{socle } Q$ and $x + y$ is a unit in Q. The proof of 4.2.4 is thus complete.

In order to make use of 4.2.4 we require the following lemma.

LEMMA 4.2.5 Suppose e and f are idempotents of Q with $\dim eQ = \dim fQ < \infty$. Then there is an element $a \in R$ such that $aQ = eQ$ and $Qa = Qf$.

Proof. Suppose $\dim eQ = n$. Then there exist sets $e_1, \ldots, e_n$ and $f_1, \ldots, f_n$ of orthogonal primitive idempotents of Q such that $e = e_1 + \ldots + e_n$ and $f = f_1 + \ldots + f_n$. Since R is a prime ring and Q is both a left and right quotient ring of R, we can choose $a_i \in e_i Q f_i \cap R$, $a_i \neq 0$, for $i = 1, \ldots, n$. Let $a = a_1 + \ldots + a_n$. Then $a \in R$ with $aQ = eQ$ and $Qa = Qf$.

PROPOSITION 4.2.6 Let I be a large right ideal of R. Then

(i) I is a prime ring and Q is a left and right quotient ring of I.

(ii) If $I + \text{socle } Q$ contains units of Q, then so does I.

Proof. (i) Suppose $a, b \in I$ are such that $aIb = 0$. If $a \neq 0$, then $Z_r(R) = 0$ implies $aI \neq 0$. Hence, since R is a prime ring, we must have $b = 0$. This shows that I is a prime ring. Now let $x \in Q$, $x \neq 0$. Then $xI \cap R \neq 0$ and so $(xI \cap R) \cap I \neq 0$, that is, $xI \cap I \neq 0$. Thus Q is a right quotient ring of I. Furthermore, since Q is also a left quotient ring of R, there exists $r \in R$ such that $0 \neq rx \in R$. By primeness of R, $Irx \neq 0$. Thus $Ix \cap I \neq 0$. Hence Q is also a left quotient ring of I.

(ii) Clearly, it suffices to consider the case $I = R$. Suppose $y \in R$, $x \in$ socle Q are such that $x + y$ is a unit of Q. Then by theorem 4.2.4, $\dim y^r = \operatorname{codim}(y) < \infty$. Choose idempotents e, $f \in Q$ such that $yQ = eQ$ and $Qy = Qf$. Then $\dim(1-e)Q = \dim(1-f)Q < \infty$. Hence by lemma 4.2.5, there exists $a \in R$ such that $aQ = (1-e)Q$ and $Qa = Q(1-f)$. Let $c = a + y$. Then $c \in R$ and c is a unit in Q. This completes the proof of 4.2.6.

Recall that when Q is not a division ring it is generated (as a ring) by its idempotents (proposition 1.2.5). The following lemma is therefore a particular case of lemma 3.2.3.

<u>LEMMA 4.2.7</u> Suppose Q is not a division ring. Let S be a subring of Q containing units of Q. Let $U = \{c \in S : c \text{ a unit in } Q\}$ and let $T = \{x \in Q : xc \in S \text{ for some } c \in U\}$. If $e \in T$ for all idempotents e of Q, then $T = Q$.

We are now in a position to give the main result of this section.

<u>THEOREM 4.2.8</u> Suppose R satisfies (A). If $R + \text{socle } Q$ is a right order in Q, then so is R.

<u>Proof</u>. If Q is a division ring, then there is nothing to prove. Suppose Q is not a division ring. In view of lemma 4.2.7, to show R is a right order in Q it will suffice to show that for each idempotent e of Q there exists a regular element $c \in R$ such that $ec \in R$ (regular elements of R are clearly units in Q). So let e be a given idempotent of Q. Let $I = (1-e)Q \cap R + eQ \cap R$. Then I is a large right ideal of R. Moreover, by proposition 4.1.2, (A) implies $IQ = Q$. Let $J = I + \text{socle } Q$. Then J is a right ideal of $R + \text{socle } Q$ and $JQ = Q$. Hence, by a standard argument, $R + \text{socle } Q$ a right order in Q implies J contains a unit of Q. By proposition 4.2.6 I contains a unit of Q, c say. Then c is a regular element of R and $ec \in R$. We are finished.

<u>Remarks</u>. (1) A closer look at the proof of 4.2.8 reveals that one needs only part of the condition (A), namely, if e is a primitive idempotent of Q then there exist $a_1, \ldots, a_n \in R$ such that $(1-e)Q = a_1Q + \ldots + a_nQ$.

(2) I strongly suspect the hypothesis of R satisfying (A) in 4.2.8 could be dropped altogether. In the case where $Q = \text{Hom}_D(V,V)$, V_D a vector space over the field D

of rational numbers, then (A) is not needed. For let e be a primitive idempotent of Q. Since $eQe \cap R$ is a nonzero subring of eQe ($\cong D$), there exists a positive integer m such that $me \in R$. Let c be a regular element of R and let $a = m(1-e)c$. Then $a \in R$ and $aQ \cong (1-e)Q$. By remark (1) it follows that R is a right order in Q.

One does not require R in 4.2.8 to satisfy (A) if R has a classical right quotient ring, that is, if R satisfies the right Ore condition. The proof of this requires two lemmas.

LEMMA 4.2.9 Let e be a primitive idempotent of Q. Then $Qe = (Qe \cap R)eQe$.

Proof. Let $x \in Qe$, $x \neq 0$. Then $(xQ \cap R)(Qe \cap R) \neq 0$ implies that there exists $d \in eQe$ such that $0 \neq xd \in Qe \cap R$. Therefore, since eQe is a division ring, $x \in (Qe \cap R)eQe$. Hence $Qe = (Qe \cap R)eQe$.

LEMMA 4.2.10 Suppose $\dim Q$ is infinite. If $R + \text{socle } Q = Q$ then $R = Q$.

Proof. Let us suppose that $Qe \not\subseteq R$ for any primitive idempotent e of Q. Then by lemma 4.2.9, for each primitive idempotent e of Q there exists $d \in eQe$ such that

$(Qe \cap R)d \not\subseteq R$. Let $\{e_i\}_{i \in I}$ be a complete set of orthogonal primitive idempotents of Q (see Chapter I, §2). For each $i \in I$ choose $d_i \in e_i Q e_i$ such that $(Qe_i \cap R)d_i \not\subseteq R$. Let x be the unique element of Q such that $xe_i = d_i$ for all $i \in I$ (see proposition 1.2.6). Then $e_i x = d_i$ for all $i \in I$. Since $R + \text{socle } Q = Q$, there exist $a \in \text{socle } Q$, $y \in R$ such that $x = a + y$. Now $a \in \text{socle } Q$ implies $e_i a = 0$ for all but a finite number of $i \in I$. Hence, since I is an infinite set, we can pick $j \in I$ such that $e_j a = 0$. But then $(Qe_j \cap R)d_j = (Qe_j \cap R)e_j x = (Qe_j \cap R)y \subseteq R$, which is a contradiction. We conclude, therefore, that there exists a primitive idempotent e of Q such that $Qe \subseteq R$.

Now let f be any other primitive idempotent of Q. Since R is a prime ring and Q is a left and right quotient ring of R, there exist $a \in eQf \cap R$ and $b \in fQe$ such that $f = ba$. Hence $f \in R$. It follows that $R \supseteq \text{socle } Q$ and hence $R = Q$. This completes the proof of 4.2.10.

<u>PROPOSITION 4.2.11</u> Suppose R has a classical right quotient ring. If $R + \text{socle } Q$ is a right order in Q, then so is R.

Proof. Let $T = \{cd^{-1}: c, \text{ regular } d \in R\}$. Then T is a subring of Q. What we have to show is that $T = Q$. Let $x \in Q$. Then there exist $a \in$ socle Q, $y \in R$ such that $a + y$ is a unit in Q and $x(a+y) \in R +$ socle Q. By the argument used in the proof of 4.2.6 (ii), there exists $b \in R \cap$ socle Q such that $b + y$ is a unit in Q. Let $d = b + y$. Then $xd \in R +$ socle Q implies that $x \in T +$ socle Q. Hence $T +$ socle $Q = Q$ and so by lemma 4.2.10, $T = Q$. The proof is complete.

For the remainder of this section we will assume Q has infinite dimension. $\bar{Q}$ will denote the ring Q/socle Q and for a subset X of Q, $\bar{X}$ will denote the image of X under the canonical mapping of Q onto $\bar{Q}$.

We now study the following problem: suppose the elements of $\bar{Q}$ can be expressed in the form $\bar{a}\bar{c}^{-1}$, $\bar{a}$ and $\bar{c}$ in $\bar{R}$ with $\bar{c}$ a unit in $\bar{Q}$. When does this imply that R is a right order in Q?

Following Barnes [1], we shall call an element a of Q a *Fredholm element* of Q if $\bar{a}$ is a unit in $\bar{Q}$, equivalently, if $\dim a^r < \infty$ and $\dim a^\ell < \infty$. The following lemma is a particular case of Barnes [1] theorem 3.2.

LEMMA 4.2.12. Let a and b be Fredholm elements of Q. Then the following hold.

(i) $\dim (ab)^r - \dim (ab)^\ell = (\dim a^r - \dim a^\ell)$
$+ (\dim b^r - \dim b^\ell)$.

(ii) $\dim (a^n b^m)^r - \dim (a^n b^m)^\ell = n(\dim a^r - \dim a^\ell)$
$+ m(\dim b^r - \dim b^\ell)$

for any positive integers m and n.

Proof. Choose $f, g \in Q$ such that $a^r = bQ \cap a^r \dotplus fQ$ and $bQ = bQ \cap a^r \dotplus gQ$. Choose $h \in Q$ such that $Q = a^r \dotplus gQ \dotplus hQ$. Then we have
$Q = bQ \cap a^r \dotplus fQ \dotplus gQ \dotplus hQ = bQ \dotplus fQ \dotplus hQ$. Hence

$$\dim (ab)^r = \dim b^r + \dim a^r \cap bQ, \text{ and}$$

$$\text{codim } (ab) = \text{codim } (a) + \dim hQ.$$

Also, $\text{codim } (b) = \dim fQ + \dim hQ$ and

$$\dim a^r = \dim fQ + \dim a^r \cap bQ.$$

By a simple manipulation of the last four equations, we obtain $\dim (ab)^r - \text{codim } (ab) = (\dim b^r - \text{codim } (b))$
$+ (\dim a^r - \text{codim } (a))$.

(i) now follows upon observing that for $x \in Q$, $\text{codim } (x) = \dim x^\ell$ if $\text{codim } (x) < \infty$. (ii) follows by induction.

LEMMA 4.2.13 Suppose R contains an element c with $\dim c^r < \dim c^\ell < \infty$ and an element d with $\dim d^\ell < \dim d^r < \infty$. Let I be a large right ideal of R. If $I + \text{socle } Q$ contains a Fredholm element of Q, then I contains a unit of Q.

<u>Proof</u>. Suppose $y \in I + \text{socle } Q$ is a Fredholm element of Q. Then $\dim y^r < \infty$ and $\dim y^\ell < \infty$. If $\dim y^r = \dim y^\ell$, then by 4.2.4 $I + \text{socle } Q$ contains a unit of Q, and hence, by 4.2.6, so does I. Suppose $\dim y^r < \dim y^\ell$. Let $n = \dim y^\ell - \dim y^r$ and let $m = \dim d^r - \dim d^\ell$. Then by lemma 4.2.12, we have $\dim (y^m d^n)^r - \dim (y^m d^n)^\ell =$ $m(\dim y^r - \dim y^\ell) + n(\dim d^r - \dim d^\ell) = m(-n) + nm = 0$, that is, $\dim (y^m d^n)^r = \dim (y^m d^n)^\ell$. Since $y^m d^n \in I + \text{socle } Q$, our previous argument shows that I contains a unit of Q. Similarly, if $\dim y^\ell < \dim y^r$ we can reach the same conclusion by using c in place of d. This completes the proof of 4.2.13.

<u>Remark</u>. An obvious example of where R does not contain elements c, d as in 4.2.13 is when $R = \text{socle } Q + \langle 1 \rangle$. A less trivial example is obtained by taking $Q = D_\infty$, D a field, and letting $R = \text{socle } Q + \langle d \rangle$ where d is the matrix

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & & \\ 0 & 0 & 1 & & & \\ 0 & 0 & 0 & 1 & & \\ \vdots & & & & \ddots & \\ & & & & & \\ & & & & & \end{pmatrix}$$

However, it is easily seen that a right order in Q must contain such elements.

PROPOSITION 4.2.14 Suppose R contains elements c and d as in 4.2.13, and suppose also that R either satisfies (A) or has a classical right quotient ring. Then R is a right order in Q if and only if the elements of $\bar{Q}$ can be expressed in the form $\bar{a}\bar{c}^{-1}$ where $\bar{a}$ and $\bar{c}$ are in $\bar{R}$ and $\bar{c}$ is a unit in $\bar{Q}$.

Proof. Suppose $\bar{Q} = \{\bar{a}\bar{c}^{-1} : \bar{a}, \bar{c} \in \bar{R}, \bar{c}$ a unit in $\bar{Q}\}$. To show R is a right order in Q it will suffice, by 4.2.8 and 4.2.11, to show that R + socle Q is a right order in Q. Hence, without loss of generality, we can suppose R contains socle Q. Let $x \in Q$. Then there exists $c \in R$ such that c is a Fredholm element of Q and $xc \in R$. Let $I = \{r \in R : xr \in R\}$. Then I is a large right ideal of R (in fact $I \supseteq$ socle Q) and $c \in I$. Hence by lemma 4.2.13, I contains a unit of Q. It is now evident that R is a right order in Q. The converse statement is trivial, so the proof is complete.

Remarks. (1) It is true that if $\bar{Q}=\{\bar{a}\bar{c}^{-1}:\bar{a}, \bar{c} \in \bar{R}, \bar{c}$ unit in $\bar{Q}\}$ then R contains either an element c with $\dim c^r < \dim c^\ell < \infty$ or an element d with $\dim d^\ell < \dim d^r < \infty$.

However I have not seen how to show that it contains both types.

(2) Notice that 4.2.14 does not say that if R is a right order in Q then $\bar{R}$ is a right order in $\bar{Q}$. Simple examples show that regular elements of $\bar{R}$ need not be units in $\bar{Q}$.

(3) We shall give examples in §3 of rings R which satisfy the hypotheses of 4.2.8 and 4.2.14 but which are not right orders in Q. Hence these results are not vacuous.

SECTION 3

We begin this section by looking at what condition (A_1) means for finite dimensional rings. We remarked in §1 that if a ring R has $Z_r(R) = 0$ and dim R_R is finite, then R satisfies (A) but not necessarily (A_1). An obvious example is obtained by choosing R so that it contains no regular elements. In the example below, R has an identity and its MRQ ring is also a left quotient ring.

EXAMPLE 1. Let D be a division ring and let $Q = D_4$. Let

$$e = \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix} \text{ and } f = \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ & & & 1 \end{pmatrix}$$

Let $R = eQ + Qf + \langle 1 \rangle$. Let $g = \begin{pmatrix} 1 & & & \\ 0 & 1 & & \\ & 1 & 0 & \\ & & 0 & 1 \end{pmatrix}$

Clearly, Q is a left and right quotient ring of R. Now $gQ \cap R \subseteq eQ + Qf$. Hence $a \in gQ \cap R$ implies $\dim aQ \leqslant 2$. Since $\dim gQ = 3$, it is clear that R does not satisfy (A_1).

The following proposition characterizes those irreducible rings R which have finite right dimension and which satisfy (A_1).

PROPOSITION 4.3.1 Suppose $\dim Q$ is finite and Q is a right quotient ring of R. Then R satisfies (A_1) if and only if $Q = \{ac^{-1} : a \in R, \text{ unit } c \in Q\}$.

Proof. Suppose R satisfies (A_1). Let $x \in Q$ be given. Choose idempotents $e, f \in Q$ such that $xQ = eQ$ and $Qx = Qf$. Then $eQ = xfQ$ implies there exists $y \in fQe$ such that $xy = e$ and $yx = f$. Since R satisfies (A_1), there exists $a \in R$ such that $aQ = eQ$. Choose an idempotent $g \in Q$ such that $Qa = Qg$. Then $yaQ = yeQ = yQ = fQ$ and $Qya = Qea = Qa = Qg$. Moreover, $fQ \cong eQ \cong gQ$. Hence, since $\dim Q$ is finite, $(1 - f)Q \cong (1 - g)Q$. Choose $k \in (1 - f)Q(1 - g)$ such that $kQ = (1 - f)Q$ and $Qk = Q(1 - g)$. Let $c = k + ya$. Then c is a unit in Q and $xc = xya = ea = a \in R$. The "only if" part is now clear.

Conversely, suppose $Q = \{ac^{-1}: a \in R, \text{ unit } c \in Q\}$. Let e be an idempotent of Q and choose a unit $c \in Q$ such that $ec \in R$. Let $a = ec$. Then $a \in R$ and $aQ = eQ$. From this it follows easily that R satisfies (A_1).

Our next proposition gives a class of finite dimensional irreducible rings which satisfy (A_1) but which are not prime rings.

<u>PROPOSITION 4.3.2</u> Suppose dim Q is finite. Let $e \neq 0$ be an idempotent of Q and let $R = eQ + Q(1-e)$. Then $Q = \{ac^{-1}: a \in R, \text{ unit } c \in Q\}$.

<u>Proof.</u> It suffices to show that for each $x \in (1-e)Q$ there exists a unit $c \in Q$ such that $xc \in Q(1-e)$. So let $x \in (1-e)Q$ be given. Since dim $Q < \infty$, we have dim $x^r \geqslant$ dim eQ. Choose an idempotent $h \in Q$ such that $hQ \subseteq x^r$ and $hQ \cong eQ$. Then $(1-h)Q \cong (1-e)Q$ because dim $Q < \infty$. Hence we can find $a \in hQe$, $b \in (1-h)Q(1-e)$ such that $aQ = hQ$, $Qa = Qe$, $bQ = (1-h)Q$ and $Qb = Q(1-e)$. Let $c = a + b$. Then c is a unit of Q and $xc = xb \in Q(1-e)$. This completes the proof.

<u>Remark.</u> If dim Q is infinite, then 4.3.2 holds if and only if dim $eQ >$ dim $(1-e)Q$. The proof of this is similar to the proof of 2.1.1. Nevertheless, R satisfies (A_1) (see example 3).

To illustrate the nature of condition (A), let us observe that a prime regular ring R which satisfies (A) must necesarily be a right self-injective ring. For let T be the MRQ ring of R. If T is a division ring then certainly R = T. If T is not a division ring then by Utumi [2] theorem 2, T is generated by its idempotents. But it is easily seen that (A) implies that R contains all the idempotents of T. Hence R = T and thus R is a right self-injective ring. Notice, however, that this is not necessarily true if R is a regular ring but not a prime ring. For example, let D be a field and F a proper subfield of D. Let I be an infinite set and let $T = \prod_{i \in I} D_i$ (ring direct product) where $D_i = D$ for all $i \in I$. Let R be the subring of T consisting of all $(x_i) \in T$ for which $x_i \in F$ for all but a finite number of $i \in I$. Then T is the MRQ ring of R and $T \neq R$. However R is a regular ring and R certainly satisfies (A) because it contains all the idempotents of T.

Let us suppose for the moment that dim Q is finite, say dim Q = n, and that Q is a right quotient ring of a prime ring R. What does primeness of R enable us to do in so far as showing R is a right order in Q? It is this. Let I be a right ideal of R. Then I can be essentially generated by a finite number of elements. Primeness of R enables us to reduce the number of generators to one.

Let us briefly recall how this is done (keeping in mind the proof of lemma 3.2.4 and also Goldie's proof in [2] theorem 3.9). We first find elements $\beta_1,\ldots,\beta_n \in R$ such that $Q = Q\beta_1 \dotplus \ldots \dotplus Q\beta_n$. Suppose $a_1,\ldots,a_k \in R$ are such that each a_iR is a uniform right ideal of R and such that the sum $a_1R + \ldots + a_kR$ is direct, and an essential submodule of I_R. By primeness of R, $a_iR\beta_i \neq 0$ for $i = 1,\ldots,k$ (note $k \leqslant n$). Choose $y_i \in R$, $i = 1,\ldots,k$, such that $a_iy_i\beta_i \neq 0$. Let $a = a_1y_1\beta_1 + \ldots + a_ky_k\beta_k$. Then $aQ = a_1Q + \ldots + a_kQ$ which implies aR is essential in I. The point to notice is that $\beta_1,\ldots,\beta_n$, once chosen, work for each right ideal I of R.

Now suppose dim Q is infinite. We make the following observation: if $\beta,\gamma \in Q$ are such that $\beta Q = \gamma Q = Q$ and $Q\beta \cap Q\gamma = 0$, then for any $x, y \in Q$, $xQ + yQ = (x\beta + y\gamma)Q$. For want of a better name, we shall refer to β and γ as a *reducing pair* for Q. However, because we are working with rings R which have Q as a right quotient ring, we prefer the following definition, which is easily seen to be equivalent to the one above for Q.

DEFINITION 4.3.3 Let R be a ring. A pair (β,γ) of elements of R is called a reducing pair for R if βR, γR and $\beta^r + \gamma^r$ are large right ideals of R.

Remarks. (1) If dim Q is finite then clearly Q cannot possess a reducing pair of elements. However when dim Q is infinite there is an abundance of reducing pairs (see example 3). If for example $Q = D_\infty$, D a division ring, the elements

$$\begin{array}{cc} \begin{array}{c} \uparrow \\ i \\ \downarrow \\ \\ \end{array} & \begin{pmatrix} 0 & 1 & & & & & & \\ 0 & 0 & 0 & 1 & & & & \\ \vdots & & & & & & & \\ 0 & \cdots & 0 & 1 & 0 & \cdots & & \\ & & & & & & & \end{pmatrix} \\ & \longleftarrow 2i \longrightarrow \end{array} \qquad \begin{array}{cc} \begin{array}{c} \uparrow \\ i \\ \downarrow \\ \\ \end{array} & \begin{pmatrix} 1 & 0 & & & & & & \\ 0 & 0 & 1 & & & & & \\ \vdots & & & & & & & \\ 0 & \cdots & 0 & 1 & 0 & \cdots & & \\ & & & & & & & \end{pmatrix} \\ & \longleftarrow 2i-1 \longrightarrow \end{array}$$

form a reducing pair for Q.

(2) If Q is a right quotient ring of R, then for $\beta,\gamma \in R$, (β,γ) is a reducing pair for R if and only if (β,γ) is a reducing pair for Q.

(3) Notice that if (β,γ) is a reducing pair of elements of Q, then β and γ generate a free multiplicative semigroup.

PROPOSITION 4.3.4 Suppose Q is a right quotient ring of R and suppose R contains a reducing pair (β,γ). If I is a right ideal of R which is essentially finitely generated, then I is essentially principally generated. Hence R satisfies (A_1) if and only if it satisfies (A).

<u>Proof</u>. Without loss of generality, we can suppose that there exist a_1, $a_2 \in I$ such that $a_1R + a_2R$ is essential in I_R. Let $a = a_1\beta + a_2\gamma$. Then $a \in I$ and $aQ = a_1Q + a_2Q$. It follows that aR is essential in I_R.

<u>PROPOSITION 4.3.5</u> Suppose dim Q is infinite and Q is a right quotient ring of R. Suppose also that $Q = \{ac^{-1}: a \in R, \text{ unit } c \in Q\}$, and that regular elements of R are units in Q. Then R is a right order in Q if and only if R possesses a reducing pair of elements.

<u>Proof</u>. Suppose R possesses a reducing pair. Let e be an idempotent of Q. Let $I = (1-e)Q \cap R + eQ \cap R$. Since R satisfies ($A_1$), by proposition 4.3.4 there exists $a \in I$ such that aR is essential in I. Hence $\ell(a,Q) = 0$. Choose $y \in Q$ such that $ay = 1$, and choose a unit $c \in Q$ such that $yc \in R$. Then $c = a(yc) \in I$. Hence $c \in R$ with $ec \in R$. By lemma 4.2.7, R is a right order in Q.

Conversely, suppose R is a right order in Q. Choose an idempotent $e \in Q$ such that $eQ \cong (1-e)Q$ ($\cong Q$). Choose $\beta \in Qe, \gamma \in Q(1-e)$ such that $\beta Q = Q$ and $\gamma Q = Q$. Then (β,γ)

is a reducing pair for Q. Now choose a regular element $c \in R$ such that βc, $\gamma c \in R$. Clearly, $(\beta c, \gamma c)$ is a reducing pair for R.

Remark. Proposition 4.3.5 when compared with 4.3.1, shows that reducing pairs for infinite dimensional rings R play a similar role to primeness for finite dimensional R, in so far as determining when $(Q \nabla R)_R$ implies R is a right order in Q. The similarity becomes even more apparent in 4.3.7.

EXAMPLE 2. Suppose dim Q is infinite and e is an idempotent of Q such that dim $(1-e)Q$ is finite $(e \neq 1)$. Let $R = eQ + Q(1-e)$. Then R satisfies the hypotheses of 4.3.5 but R is not a right order in Q (corollary 2.1.2). Hence R does not possess a reducing pair. Admittedly, this is not a good example because R is not a prime ring either.

By a slight modification of the proof of lemma 3.2.3, we can obtain the following

LEMMA 4.3.6 Suppose Q is not a division ring. Let R be a subring of Q. Let $U = \{a \in R\colon a$ right invertible in $Q\}$ and let $T = \{x \in Q\colon xa \in R$ for some $a \in U\}$. If $e \in T$ for all idempotents e of Q, then $T = Q$.

We come now to the principal result of this section.

THEOREM 4.3.7 Suppose Q is a right quotient ring of R, $\dim Q$ infinite. If R satisfies (A) and contains a reducing pair, then for each $x \in Q$ there exists $a \in R$ such that a has a right inverse in Q and $xa \in R$. The converse is also true.

Proof. Let e be an idempotent of Q and let $I = (1-e)Q \cap R + eQ \cap R$. Then I is a large right ideal of R and (A) implies I is essentially finitely generated. Hence, by proposition 4.3.4, there exists $a \in I$ such that aR is an essential submodule of I_R. Since $Z(Q_R) = 0$, we have $\ell(a,Q) = 0$. Hence a has a right inverse in Q. Furthermore, $a \in I$ implies $ea \in R$. All there remains to do now in order to establish the first part of 4.3.7 is to apply lemma 4.3.6.

The converse statement is easily shown.

The following corollary says that for R and Q as in 4.3.7, we can obtain much the same sort of information concerning the relative right ideal structures of R and Q as when R is a right order in Q. The proofs are identical to the corresponding proofs for right orders (see, for example, Faith [1] p.79).

<u>COROLLARY 4.3.8</u> Suppose Q is a right quotient ring of R and suppose R satisfies (A) and contains a reducing pair. Then the following hold.

(i) For $x_1,\dots,x_n \in Q$ there exists $a \in R$ with a right invertible in Q and such that $x_i a \in R$ for $i = 1,\dots,n$.

(ii) If I is a right ideal of R then $IQ = \{ay : a \in I,\ y \in Q,\ y^r = 0\}$.

(iii) If I and J are right ideals of R then $(I \cap J)Q = IQ \cap JQ$.

We conclude this section by looking at two examples.

<u>EXAMPLE 3.</u> Suppose $\dim Q$ is infinite and e is an idempotent of Q such that $Q \cong eQ$. If R is a subring of Q containing Qe then R satisfies (A) and contains a reducing pair. To see this, write $e = f + g$ where f and g are orthogonal idempotents with $\dim fQ = \dim gQ$. Then $fQ \cong Q$ and $gQ \cong Q$. Choose $\beta \in Qf$, $\gamma \in Qg$ such that $\beta Q = \gamma Q = Q$. Then (β,γ) is a reducing pair for R. Now let B be a right ideal of Q. Then, since $QeQ = Q$, we have $(B \cap R)Q \supseteq (BQe)Q = BQ = B$. Hence by 4.1.2, R satisfies (A) (and hence, by 4.3.4, R satisfies (A_1)).

If R has an identity, then $R \supseteq eQe + \langle 1 - e \rangle$. Hence R contains elements c, d with $\dim c^r < \dim c^\ell < \infty$ and $\dim d^\ell < \dim d^r < \infty$ (c.f. 4.2.14).

Notice, however, that if $R \neq Q$ then R can never be a right order in Q. For suppose R is a right order in Q. Then $QeRQ = Q$ implies QeR contains a unit of Q. Thus $QeR = Q$, and hence $R = Q$.

<u>EXAMPLE 4</u>. Again we suppose dim Q is infinite. Let Y be a maximal right ideal of Q and let $R = \{x \in Q: xY \subseteq Y\}$. Then R cannot possess a reducing pair. For suppose (β,γ) is a reducing pair for R. Then $\beta Q = \gamma Q = Q$ and $Q\beta \cap Q\gamma = 0$. Choose orthogonal idempotents $f, g \in Q$ such that $Q\beta = Qf$ and $Q\gamma = Qg$. Since $\beta fQ = Q$ and $\gamma gQ = Q$ there exist $\alpha \in fQ$, $\eta \in gQ$ such $\alpha\beta = f$, $\beta\alpha = 1$, $\eta\gamma = g$ and and $\gamma\eta = 1$. If $Y + \alpha Y = Q$, then $Q = \beta Q = \beta Y + Y \subseteq Y$, a contradiction. Hence, since Y is a maximal right ideal of Q, we have $\alpha Y \subseteq Y$ and hence $\alpha \in R$. Thus $f = \alpha\beta \in R$. If $fY = fQ$, then $Y \supseteq \beta(fY) = \beta fQ = Q$, a contradiction. Thus $fY \neq fQ$. Similarly, we can show that $g \in R$ and $gY \neq gQ$. But now we have $Y = fY + (1-f)Y \subsetneq fQ + (1-f)Y \subsetneq Q$ which contradicts Y being a maximal right ideal of Q. The conclusion, therefore, is that R cannot possess a reducing pair of elements. Notice, however, that there exist reducing pairs (β,γ) for Q with β in R. Also, Q is a right quotient ring of R and R can be chosen such that R and R modulo the maximum ideal of Q are prime rings.

SECTION 4

In this section we assume Q is a right quotient ring of a subring R with identity (which is necessarily the identity of Q). In the terminology of Findlay [1], if S is a ring with identity then a *left-flat epimorphic extension* of S is a ring T together with a ring homomorphism $\phi: S \to T$ such that ϕ is both a monomorphism and an epimorphism in the category of rings, and such that ϕ induces on T the structure of a flat left S-module. The reader is cautioned, however, that this situation is perhaps more commonly described by saying that the homomorphism $\phi: S \to T$ is a "right" flat bimorphism. The existence of a unique (up to isomorphism over S) maximal left-flat epimorphic extension of S has been established by Popescu and Spircu [1], Findlay [1] and Morita [2]. In general, the maximal left-flat epimorphic extension P(R) of R appears as a proper subring of Q (where ϕ is the canonical injection of R into P(R)). In this section we briefly indicate the role played by condition (A) on R in determining when P(R) = Q, that is, when Q is a left-flat epimorphic extension of R.

Cateforis in [2] theorem 1.6 showed, in particular, the equivalence of the following two statements.

(a) For each $q \in Q$, $(R:q) = \{x \in R: qx \in R\}$ is an essentially finitely generated right ideal of R, equivalently, $(R:q)Q = Q$.

(b) $Z(Q \otimes_R Q)_R = 0$ and ${}_RQ$ is flat.

Now statement (b) is equivalent to Q being a left-flat epimorphic extension of R. Thus, if R satisfies (A) and contains a reducing pair of elements then, by 4.3.7, Q is a left-flat epimorphic extension of R (it would, however, be easier to prove this directly using 4.3.8). To say R satisfies (A) is, of course, equivalent to saying that (a) holds for all the idempotents of Q.

Each of (a) and (b) is equivalent in turn to

(c) R satisfies (A) and ${}_RQ$ is flat.

(d) R satisfies (A) and $(I \cap J)Q = IQ \cap JQ$ for all right ideals I, J of R.

This equivalence is essentially contained in Cateforis [2] theorem 1.6 and Cateforis [1] theorem 2.1. Notice that the latter result of Cateforis gives an internal characterization of R for ${}_RQ$ to be flat.

<u>Remarks</u>. (1) For the general theory of flat epimorphic extensions see Silver [1], Popescu and Spircu [1], Findlay [1], Morita [1], [2], [3], and Tachikawa [1].

(2) If dim Q is finite then Q is a left-flat epimorphic extension of R.

(3) Statement (a) is easily seen to be equivalent to R being a right quasi-order of Q in the sense of Popescu and Spulber [1].

(4) Morita [1] has shown that any left-flat epimorphic extension (in Findlay's sense) of a ring S with identity can be realized as the double centralizer of a certain injective module V_S. See his paper for the precise details. This ties in beautifully with Lambek's characterization of the Utumi maximal right quotient ring of S as the double centralizer of the injective hull of S_S (Lambek [1]).

CHAPTER V

THE IDEAL STRUCTURE OF A RIGHT ORDER

This chapter consists of a rather loosely connected collection of results and problems on some aspects of the ideal structure of a right order in a left full linear ring Q. In §1 we give two (internal) characterizations of a ring R which is a right order in Q. Neither can be regarded as complete or final. In §2 we begin with a simple example of a right order R in D_∞, D the field of rational numbers, such that R does not admit a Faith-Utumi description in the strictest sense but which nevertheless satisfies a property, called (B), which permits a description of R in terms of subrings of D. If property (B) were shared by all right orders in left full linear rings of countable dimension this would mean, for example, that there are no proper right orders in D_∞ for a finite field D. Whereas I have not constructed such right orders, it is not at all obvious that they cannot exist. On the contrary, their existence is suggested by the existence of certain right orders in the MRQ ring of D_∞/socle D_∞. We consider these and related questions in §2.

SECTION 1

Our first characterization of a right order R in Q, dim Q infinite, is a fairly obvious consequence of theorem 4.3.7.

PROPOSITION 5.1.1 A ring R is a right order in a left full linear ring of right dimension $\aleph \geq \aleph_0$ if and only if R satisfies the following four conditions.

(i) R is an irreducible ring containing uniform right ideals and dim $R_R = \aleph$.

(ii) The closed right ideals of R are right annihilator ideals and each such right ideal is an essential extension of a finitely generated right ideal.

(iii) R possesses a reducing pair of elements.

(iv) For each $a \in R$ with $a^{\ell} = 0$, aR contains regular elements of R.

Proof. Suppose R is a right order in a left full linear ring Q where dim $Q = \aleph \geq \aleph_0$. Then R has zero right singular ideal and by proposition 1.2.2, R is an irreducible ring containing uniform right ideals. By the lattice isomorphism of 1.1.3 we have dim R_R = dim $Q_Q = \aleph$. Thus (i) is established. By 3.3.4, Q is left intrinsic over R and thus, by theorem 3.1.2, the closed right ideals of R are right annihilator ideals. The second part of (ii), which is condition (A) of chapter IV, was established in 4.1.4. (iii) was shown in proposition 4.3.5. Finally, (iv) follows upon observing that for $a \in R$ with $\ell(a,R) = 0$ then $\ell(a,Q) = 0$ since Q is left intrinsic over R. Thus $aQ = Q$ and hence $ab = 1$ for some $b \in Q$. Since R is a right order in Q there exists a regular element $c \in R$ such that $bc \in R$. Then $c \in aR$ and this gives (iv).

Conversely, suppose R satisfies (i)-(iv). Then by 1.2.2, condition (i) implies that the MRQ ring of R is a left full linear ring, Q say. By the lattice isomorphism of 1.1.3, $\dim Q = \dim R_R = \aleph$. Moreover, by theorem 3.1.2, Q is left intrinsic over R and thus a regular element of R is a unit of Q. Now let $x \in Q$ be given. By theorem 4.3.7, (ii) and (iii) imply that there exists $a \in R$ with $a^\ell = 0$ and such that $xa \in R$. By (iv) there is a regular element c of R in aR. Then $xc \in R$. Thus R is a right order in Q and the proof of 5.1.1 is complete.

<u>Remark.</u> If $\aleph = \aleph_0$ then in (i) "irreducible ring" can be replaced by "prime ring with zero right singular ideal". See corollary 2.2.3.

We can illustrate very simply that the conditions (i)-(iv) of 5.1.1 are independent. Trivially, (i) is not a consequence of (ii), (iii) and (iv). For an example of a ring R which satisfies (i), (iii) and (iv) but not (ii), choose R to be a right full linear ring with dim ${}_R R$ infinite. Example 2 of chapter IV, §3, shows (iii) is not a consequence of (i), (ii) and (iv). Finally, example 3 of chapter IV, §3, gives rings which satisfy (i), (ii) and (iii) but not (iv).

Our second characterization of a right order R in Q, dim Q infinite, is in terms of the right annihilator ideals of R which have the same dimension as R. In the case of Q itself, these are the right ideals which are isomorphic to Q. For a right order R, they are the right ideals which are closures (in $L(R_R)$) of right ideals isomorphic to R.

PROPOSITION 5.1.2 A ring R is a right order in a left full linear ring of right dimension $\aleph \geq \aleph_0$ if and only if R satisfies the following conditions.

(i) R is an irreducible ring containing uniform right ideals and dim $R_R = \aleph$.

(ii) The closed right ideals of R are right annihilator ideals and if B is a right annihilator ideal with dim B_R = dim R_R then B has the form $b^{\ell r}$ for some $b \in R$ with $b^r = 0$.

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(iii) R possesses a reducing pair of elements.

<u>Proof</u>. Suppose R is a right order in a left full linear ring Q where dim Q = $\aleph \geq \aleph_0$. By 5.1.1, R satisfies (i) and (iii) and its closed right ideals are right annihilator ideals. Now let B be a right annihilator ideal of R with dim B_R = dim R_R. Then there exists an idempotent e of Q such that $B = eQ \cap R$. By 1.1.3 dim eQ = dim Q and thus, by 1.2.3, $Q \cong eQ$. Hence there is an element $y \in Q$ with $y^r = 0$ and $yQ = eQ$. Since R is a right order in Q, we can find a regular element $c \in R$ such that $yc \in R$. Let $b = yc$. Then $b^r = 0$ and $bQ = eQ$. Hence bR is essential in B_R and thus by 1.1.2, $b^\ell = (bR)^\ell = B^\ell$ where annihilators are taken in R. Since B is a right annihilator ideal of R, we have $B = B^{\ell r} = b^{\ell r}$. This establishes (ii).

Conversely, suppose R satisfies (i), (ii) and (iii). Then, as in the proof of 5.1.1, the MRQ ring of R is a left full linear ring, Q say, with dim Q = $\aleph$. What we have to show is that R is a right order in Q. By the argument used in 5.1.1, regular elements of R are units in Q. Since Q is generated (as a ring) by the idempotents e for which $eQ \cong (1-e)Q$ (proposition 1.2.5), to complete the proof it will suffice, by 3.2.3, to show that for each such idempotent e there exists a regular element $c \in R$

such that $ec \in R$. So let e be given. Let $I = (1-e)Q \cap R + eQ \cap R$. Since $(1-e)Q \cap R$ and $eQ \cap R$ are closed right ideals of R and have the same dimension as R_R, (ii) implies that there exist $a, k \in R$ such that $(1-e)Q \cap R = k^{\ell r}$ and $eQ \cap R = a^{\ell r}$, annihilators taken in R. By 1.1.2, together with the fact that closed right ideals of R are right annihilator ideals, we have kR essential in $(1-e)Q \cap R$ and aR essential in $eQ \cap R$. Since R possesses a reducing pair of elements, there exists $y \in I$ such that yR is essential in I. Moreover, as I is essential in R, we must have $\ell(y,Q) = 0$, equivalently, $yQ = Q$. Choose an idempotent $f \in Q$ such that $Qy = Qf$. Then $fQ \cong Q$. Let $B = fQ \cap R$. Then $\dim B_R = \dim R_R$ and hence (ii) implies there exists $b \in B$ with $b^r = 0$ and $bQ = fQ$. Let $c = yb$. Then $cQ = Qc = Q$. Thus c is a regular element of R. Furthermore, $ec \in R$. We are finished.

Remarks. (1) Suppose $\dim Q$ is infinite, and let e be an idempotent of Q such that $eQ \cong (1-e)Q$. Let $R = eQ + Q(1-e) + \text{socle } Q$. Then R satisfies (i) and (iii) of 5.1.2. Also, the closed right ideals of R are of the form $b^{\ell r}$, $b \in R$. However, R does not satisfy (ii) of 5.1.2. For choose $\delta \in (1-e)Qe$ such that $\delta Q = (1-e)Q$, $Q\delta = Qe$. Let $f = e + \delta$ and let $B = fQ \cap R$. Then B is a right annihilator ideal of R with $\dim B_R = \dim R_R$. But as $B \subseteq Q(1-e) \cap \text{socle } Q$, there is no element $b \in R$ for which $B = b^{\ell r}$, $b^r = 0$. This illustrates the nature of condition (ii) of 5.1.2.

(2) A somewhat different characterization of a right order in Q is contained in Gupta [1] theorem 3.17. Harada [1] has studied rings which are left and right orders in a prime ring with nonzero socle.

For the remainder of the chapter we will be interested mainly in the case when dim Q is countable.

The example below shows that for a right order R in Q and for an idempotent e of Q, $eQe \cap R$ need not be a right order in eQe, even if R contains the socle of Q. It also shows that in contrast to the left annihilator ideals of R, which we know are of the form a^ℓ, $a \in R$ (see 4.1.4), the right annihilator ideals of R need not have the form a^r, $a \in R$. Incidentally, this is the case if dim Q is finite (see Goldie [2] theorem 3.7).

EXAMPLE 1. Let K be a right Ore domain which is not a left Ore domain. Let D be the right quotient division ring of K. Let $T = D_\infty$ and $A = K_\infty + \text{socle } D_\infty$. Choose $d \in D$, $d \neq 0$, such that $Kd \cap K = 0$ and let $y \in T$ be the scalar matrix corresponding to d. Then $Ay \cap A \subseteq \text{socle } D_\infty$. Now let $Q = T_2$ and $R = A_2$. Then R is a right order in Q.

Let

$$e = \begin{pmatrix} 1 & y \\ 0 & 0 \end{pmatrix}$$

Then $Qe \cap R \subseteq$ socle Q, so certainly $eQe \cap R$ is not a right order in eQe. Let $B = (1-e)Q \cap R$. Then B is a right annihilator ideal of R. Suppose there exists $a \in R$ such that $B = r(a,R)$. Then $r(a,Q) = (1-e)Q$ and this implies $Qa = Qe$, which contradicts $Qe \cap R \subseteq$ socle Q. Hence B is not of the form a^r, $a \in R$.

However, when R is both a left and right order in Q the situation is quite different, as we now show.

<u>PROPOSITION 5.1.3</u> Suppose $\dim Q = \aleph_0$ and R is a left and right order in Q. Then for any nonzero idempotent e of Q, $eQe \cap R$ is a left and right order in eQe. Moreover, the right annihilator ideals of R are of the form a^r, $a \in R$.

<u>Proof</u>. Let e be a nonzero idempotent of Q. If $e \in$ socle Q then, as R is a prime ring (corollary 2.2.3) with Q a left and right quotient ring of R, $eQe \cap R$ is a right order in eQe (see, for example, Utumi [5] theorem 3.2). If $e \notin$ socle Q then $Q \cong eQ$ and there exists $b \in R$ such that $eQ = bQ$ and $b^r = 0$. Then there exists $a \in Qe$ such that $ba = e$ and $ab = 1$. The map $x \to bxa$, $x \in Q$, is a ring isomorphism of Q onto eQe. Let $d \in eQe$ be given.

Then $d = bxa$ where $x = adb$. Choose regular elements c_1, c_2 of R such that $xc_1 \in R$ and $c_2 a \in R$. Let $c = c_1 c_2$. Then bca is a regular element of $eQe \cap R$ and $d(bca) = b(xc_1 c_2)a \in eQe \cap R$. Similarly, there exists a regular element $k \in eQe \cap R$ such that $kd \in eQe \cap R$. Hence $eQe \cap R$ is a left and right order in eQe.

Now let B be a closed right ideal of R. Then $B = fQ \cap R$ for some idempotent f of Q. Choose a regular element $c \in R$ such that $cf \in R$. Let $a = c(1-f)$. Then $a \in R$ and $B = r(a,R)$.

EXAMPLE 2. Suppose $\dim Q = \aleph_0$. Let $\bar{Q} = Q/\text{socle } Q$ and for a subset X of Q let $\bar{X}$ denote the image of X under the canonical map of Q onto $\bar{Q}$. Since $\bar{Q}$ is a simple ring with identity but is not Artinian, there exists a left ideal Y of Q such that $Y \supseteq \text{socle } Q$ and $\bar{Y}$ is a (proper) large left ideal of $\bar{Q}$. Let R be any subring of Q which contains Y. Then R satisfies (A) (see 4.1.1 for definition) and contains a reducing pair of elements (see example 3, chapter IV, §3). Moreover, $\bar{R}$ is a prime ring and $\bar{Q}$ is a left and right quotient ring of $\bar{R}$. However, as remarked earlier, unless $R = Q$, R is not a right order in Q. This simple example suggests that even when $\dim Q$ is countable, primeness of R modulo the maximum ideal of Q does not play a very decisive role in determining when

R is a right order in Q.

SECTION 2

Let D be the field of rational numbers and Z the ring of (rational) integers. Let $Q = D_\infty$ and $R = \text{socle } D_\infty + (Z_\infty)d$ where d is the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & \\ 0 & 2 & & & & \\ 0 & & 1 & & & \\ 0 & & & 3 & & \\ \vdots & & & & 1 & \\ & & & & & \ddots \end{pmatrix}$$

Then R is a right order in Q but there does not exist a right order K in D and a subring U of R such that $U \cong K_\infty$ and such that R is a right quotient ring of U. For otherwise R would contain nonzero elements from the centre of Q, and this is clearly not the case. Of course, R contains a subring which is isomorphic to Z_∞ but as R is not a right quotient ring of it, this is of little consequence. Hence R does not admit a Faith-Utumi description in the strictest sense (c.f. Faith and Utumi [2]). Nevertheless, R has the following property:

There exist a complete set $\{e_i\}_{i\in I}$ of orthogonal primitive idempotents of Q and subrings $\{D_i\}_{i\in I}$ of R such that

(a) $(Qe_i \nabla D_i)_{D_i}$ for all $i \in I$, and

(b) R contains $\prod_{i\in I} D_i = \{x \in Q: xe_i \in D_i, \forall i \in I\}$.

We make the following definition.

<u>DEFINITION 5.2.1</u> Let R be an irreducible ring containing uniform right ideals and let Q be the MRQ ring of R. R is said to satisfy (B) if there exists a complete set $\{e_i\}_{i\in I}$ of orthogonal primitive idempotents of Q and subrings $D_i \subseteq Qe_i \cap R$, $i \in I$, such that conditions (a) and (b) above are satisfied.

<u>Remarks</u>. (1) If R satisfies (B) then there is no loss of generality in assuming $e_iD_i \subseteq D_i$ for all $i \in I$. Simply replace D_i by $(1-e_i)Q \cap D_i + e_iQ \cap D_i$. Then, letting $D_{ii} = e_iD_i$, we have $(Qe_i \nabla D_i)_{D_{ii}}$ for all $i \in I$. Notice that D_{ii} is a right order in e_iQe_i for all $i \in I$. Hence if $Q = \mathrm{Hom}_D(V,V)$, V_D a vector space over a finite field D,

then the only subring of Q which satisfies (B) is Q itself.

(2) If R satisfies (B) then clearly R is a prime ring. Thus if dim Q is uncountable, then not every right order in Q satisfies (B).

(3) If R satisfies (B) then $R + \text{socle } Q$ is a right order in Q.

(4) If $\dim Q = n < \infty$ and R is a right order in Q then R satisfies (B) for any set $\{e_i\}_1^n$ of orthogonal primitive idempotents of Q for which $Qe_i \cap R \neq 0$ for $i = 1,\ldots,n$. Simply let $D_i = Qe_i \cap R$.

(5) If R satisfies (B) and $c, d \in R$ are units in Q, then cRd satisfies (B).

The following property of a ring R, $\text{socle } Q \subseteq R \subseteq Q$, which satisfies (B) seems worth recording.

PROPOSITION 5.2.2 Suppose dim Q is countable and R is a subring of Q which contains the socle of Q. If R satisfies (B), then for each countable set $\{x_i\}_1^\infty \subseteq Q$ there exists a regular element $d \in R$ such that $x_i d \in R$ for $i = 1,2,3,\ldots$.

Proof. Let N be the set of natural numbers. Suppose $\{e_i\}_{i\in N}$ is a complete set of orthogonal primitive idempotents of Q and $\{D_i\}_{i\in N}$ subrings of R such that

$(Qe_i \nabla D_i)_{D_i}$ for all $i \in N$, and such that $\{x \in Q\colon xe_i \in D_i, \forall i \in I\} \subseteq R$. We can suppose $e_i D_i \subseteq D_i$ for all $i \in N$, so letting $D_{ii} = e_i D_i$ we have $(Qe_i \nabla D_i)_{D_{ii}}$. Let $\{x_i\}_{i \in N}$ be given. For each $i \in N$ choose $d_i \in D_{ii}$, $d_i \neq 0$, such that $(x_j e_i) d_i \in D_i$ for $j = 1, \ldots, i$. Let d be the unique element of Q satisfying $de_i = d_i$ for all $i \in N$ (see 1.2.6). Then $d \in R$. Furthermore, for $n \in N$, $(x_n d) e_i \in D_i$ for all $i \geqslant n$. Hence for each $n \in N$, we have $(x_n d)(1 - \sum_{j=1}^{n} e_j) e_i \in D_i$ for all $i \in N$. This implies $(x_n d)(1 - \sum_{j=1}^{n} e_j) \in R$ for all $n \in N$. Since socle $Q \subseteq R$, we have $x_n d \in R$ for all $n \in N$.

PROPOSITION 5.2.3 Suppose dim Q is infinite and R is a right order in Q with the property: for each countable set $\{x_i\}_1^\infty \subseteq Q$ there exists a regular element $d \in R$ such that $x_i d \in R$ for $i = 1, 2, 3 \ldots$. Then R contains a right ideal I which is also a right order in Q and whose finitely generated right ideals are principal.

Proof. Choose a reducing pair (β, γ) of elements of R. Then $\beta Q = \gamma Q = Q$ and $Q\beta \cap Q\gamma = 0$. Choose orthogonal idempotents f and g of Q such that $Q\beta = Qf$ and $Q\gamma = Qg$. Then there exist $\alpha \in fQ, \delta \in gQ$ such that $\beta\alpha = 1$, $\alpha\beta = f$, $\gamma\delta = 1$, $\delta\gamma = g$. Let $I = \{a \in R\colon \langle \alpha, \delta \rangle a \subseteq R\}$ where $\langle \alpha, \delta \rangle$ is the subring of Q generated by α and δ.

Then I is a right ideal of R and since $|\langle \alpha,\delta \rangle| = \aleph_0$, I contains a regular element of R. Hence I is a right order in Q. Since $\alpha I \subseteq I$ and $\delta I \subseteq I$, for $a, b, c, d \in I$ we have $ac + bd = (a\beta + b\gamma)(\alpha c + \delta d) \in (a\beta + b\gamma)I$. Thus $aI + bI = (a\beta + b\gamma)I$ for any $a,b \in I$. It is now clear that finitely generated right ideals of I are principal.

For a complete set $\{e_i\}_{i\in I}$ of orthogonal primitive idempotents of Q and for a subset F of I, let e_F be the unique element of Q for which $e_F e_i = e_i$ for all $i \in F$, $e_F e_i = 0$ for all $i \notin F$. Clearly, e_F is an idempotent of Q and e_F commutes with each e_i. The ring R of example 1, §1, satisfies (B), but not with respect to just any complete set of orthogonal primitive idempotents of its MRQ ring Q. For example, if $\{e_i\}_{i\in N}$ is a complete set for which $e = e_F$ for some $F \subseteq N$, then (B) does not hold with respect to this set because $Qe \cap R \subseteq$ socle Q. An obvious necessary condition for a right order R in Q to satisfy (B) with respect to a complete set $\{e_i\}_{i\in I}$ of orthogonal primitive idempotents of Q is that there exists a regular element c of R such that $ce_F \in R$ for all subsets $F \subseteq I$.

<u>QUESTION 1</u>. If dim Q is countable, which right orders in Q satisfy (B)?

Henceforth we assume dim Q is countable. $\bar{Q}$ will denote the ring Q/socle Q and for a subset X of Q, $\bar{X}$ will denote the image of X under the canonical map of Q onto $\bar{Q}$. Suppose R is a right order in Q. If I is a right ideal of R, $\bar{I} \neq \bar{0}$, then I contains an element b with $b^r = 0$. This is easily seen upon observing that for $x \in Q$, $\bar{x} \neq \bar{0}$, there exist y, $z \in Q$ such that $yxz = 1$. In particular, if I is a two-sided ideal of R with $\bar{I} \neq \bar{0}$ then I contains an element b with $b^r = 0$. But must I contain a regular element of R? Equivalently, must $IQ = Q$? Suppose not and let $Y = IQ$. Then $R \subseteq N(Y,Q) = \{x \in Q: xY \subseteq Y\} \subsetneq Q$. Hence N(Y,Q) is a (proper) right order in Q. Is this possible? (N(Y,Q) is called the "normalizer" of Y in Q.)

QUESTION 2. If dim Q is countable, does there exist a proper right ideal Y of Q, $Y \supseteq$ socle Q, $\bar{Y}$ essential in $\bar{Q}$, such that $N(Y,Q) = \{x \in Q: xY \subseteq Y\}$ is a right order in Q?

The reason for requiring $\bar{Y}$ to be essential in $\bar{Q}$ is that Y is an ideal of N(Y,Q), and if the latter is a right order in Q then N(Y,Q)/socle Q is a prime ring (proposition 2.2.6). Thus, if $\bar{Y} \neq \bar{0}$, $\bar{Y}$ must be essential in N(Y,Q)/socle Q and hence in $\bar{Q}$. Notice that not just any such right ideal Y would do (for example, Y maximal). But it seems reasonable that if N(Y,Q) possesses a reducing pair of elements then N(Y,Q) is a right order in Q. Notice that $N(Y,Q) \cong \mathrm{Hom}_Q(Y,Y)$.

We could equally as well have stated question 2 as follows: let $A = Q/\text{socle } Q$. Does there exist a proper right ideal J of A such that $N(J,A) = \{x \in A: xJ \subseteq J\}$ is a right order in A? Again, J would have to be a large right ideal of A. Since A is a simple ring with identity, its MRQ ring, T say, is a simple right self-injective ring with identity. By Osofsky [1], $A \neq T$. Actually T is not even left intrinsic over A, since there exist right ideals of A which have zero left annihilator in A but which are not essential in A. The interesting thing, however, is that if $Q = D_\infty$ and D is not too big ($|D| \leqslant 2^{\aleph_0}$ will do), then there exist right ideals K of T for which $N(K,T)$ is a right order in T. We next prove a slightly more general result.

<u>PROPOSITION 5.2.4</u> Let D be a division ring and let $A = D_\infty/\text{socle } D_\infty$. Let T be the MRQ ring of A. Suppose A contains a family of independent nonzero right ideals such that the cardinality $\aleph$ of the family is the maximum cardinality such families can have. Let $\{e_\alpha A\}_{\alpha \in \Omega}$ be a family of independent nonzero principal right ideals of A with $|\Omega| = \aleph$ and such that $\sum_\Omega e_\alpha A$ is a large right ideal of A. Let $J = \sum_\Omega e_\alpha A$ and let $S = \{x \in T: xJ \subseteq J\}$. Then $T = \{bc^{-1}: b, c \in S \text{ with } c \text{ a unit in } T\}$.

Remarks. (1) $S \cong \mathrm{Hom}_A(J,J) \cong$ ring of all $\aleph \times \aleph$ column-finite matrices over A. (J_A is a free A-module on $\aleph$ generators.)

(2) If $|D| \leqslant 2^{\aleph_0}$ then $|A| = 2^{\aleph_0}$. Hence, since we can always construct $2^{\aleph_0}$ orthogonal idempotents of A (see Osofsky [1]), A satisfies the hypothesis of the proposition.

Proof. Firstly, observe that each nonzero principal right ideal of A is isomorphic to A (as a right A-module). Furthermore, we make the following

Claim. For any nonzero right ideal I of A, there exists a family $\{f_\gamma A\}_{\gamma\in\Gamma}$ of independent nonzero principal right ideals of A such that $|\Gamma| = |\Omega|$ and $\sum_\Gamma f_\gamma A$ is essential in I_A.

To verify this claim, we choose (by Zorn's lemma) an independent family $\{f_\gamma A\}_{\gamma\in\Gamma}$ such that $\sum_\Gamma f_\gamma A$ is essential in I_A. By hypothesis, $|\Gamma| \leqslant |\Omega|$. If $|\Gamma| < |\Omega|$, we pick $\gamma_0 \in \Gamma$ and using an isomorphism of A onto $f_{\gamma_0}A$ we can obtain an independent family $\{g_\alpha A\}_{\alpha\in\Omega}$ such that $\sum_{\alpha\in\Omega} g_\alpha A$ is essential in $f_{\gamma_0}A$. Then the family $\{g_\alpha A\}_{\alpha\in\Omega} \cup \{f_\gamma A\}_{\gamma\in\Gamma\setminus\{\gamma_0\}}$ has cardinality $|\Omega|$ and the desired property. This establishes the claim.

Now let $t \in T$ be given. Since J_A is essential in T_A, there exists a large right ideal I of A, $I \subseteq J$, such that $tI \subseteq J$. Choose an independent family $\{f_\alpha A\}_{\alpha\in\Omega}$ of nonzero principal right ideals of A such that $\sum_{\Omega} f_\alpha A$ is essential in I_A. Since $e_\alpha A \cong f_\alpha A$ for all $\alpha \in \Omega$, we can construct an isomorphism Ψ of J_A onto $\sum_{\Omega} f_\alpha A$. Let $c \in T$ induce Ψ by its left multiplication (T_A is injective). Then c is a unit of T and $c \in S$. Moreover, $(tc)J = t(cJ) \subseteq tI \subseteq J$. Hence $tc \in S$. This completes the proof of 5.2.4.

Remark. If $K = \sum_{\Omega} e_\alpha T$ then $S \subseteq N(K,T) \subsetneq T$. Hence $N(K,T)$ is a proper right order in T (regular elements of $N(K,T)$ are certainly units in T).

We shall call a set $\{\xi_i\}_{i \in I}$ of elements of Q a *Q-independent set* if for $\{i_1,\ldots,i_n\} \subseteq I$ and $x_{i_1},\ldots,x_{i_n} \in Q$, $\xi_{i_1}x_{i_1} + \ldots + \xi_{i_n}x_{i_n} = 0$ implies $x_{i_1} = \ldots = x_{i_n} = 0$, or equivalently, if $\xi_i^r = 0$ for all $i \in I$ and the sum $\sum_I \xi_i Q$ is direct.

Let $\{e_i\}_1^\infty$ be a set of orthogonal idempotents of Q such that $Q \cong e_i Q$ for $i = 1,2,3\ldots$, and such that socle $Q \subseteq \sum_1^\infty e_i Q$. For each i choose $\xi_i \in e_i Q$, $\eta_i \in Qe_i$ such that $\xi_i \eta_i = e_i$, $\eta_i \xi_i = 1$. Then $\{\xi_i\}_i^\infty$ is a maximal Q-independent set of elements of Q. Now let $\{x_i\}_1^\infty \subseteq Q$ be given. Let x be the unique element of Q such that $xe_i = x_i \eta_i$ for $i = 1,2,3\ldots$ (see proposition 1.2.6). Then $x\xi_i = x_i$ for $i = 1,2,3\ldots$. Hence the "only if" part of the following proposition is clear.

PROPOSITION 5.2.5 Suppose socle $Q \subseteq R \subseteq Q$ and R possesses a reducing pair of elements. Suppose also that R contains elements c, d with $\dim c^r < \dim c^\ell < \infty$ and $\dim d^\ell < \dim d^r < \infty$. Then R is a right order in Q if and only if for each countable set $\{x_i\}_1^\infty \subseteq Q$ there exist a maximal Q-independent set $\{\xi_i\}_1^\infty \subseteq Q$ and an element $a \in R$ such that

$$a\xi_i = x_i \quad \text{for} \quad i = 1,2,3\ldots .$$

Proof. We need only establish the "if part". Let $x \in Q$ be given. Choose a Q-independent set $\{\alpha_i\}_1^\infty \subseteq Q$ such that socle $Q \subseteq \sum_1^\infty \alpha_i Q$ and let $x_i = x\alpha_i$. By hypothesis, there exist a maximal Q-independent set $\{\xi_i\}_1^\infty \subseteq Q$ and an element $a \in R$ such that $a\xi_i = x_i$ for $i = 1,2,3\ldots$. Since for each i, the map $\alpha_i y \to \xi_i y$ is an isomorphism of $\alpha_i Q$ onto $\xi_i Q$, there exists $b \in Q$ such that $b^r = 0$ and $b\alpha_i = \xi_i$ for $i = 1,2,3\ldots$. Then $ab\alpha_i = a\xi_i = x_i = x\alpha_i$ for all i. Since $\sum_1^\infty \alpha_i Q \supseteq$ socle Q, we have $ab = x$. Moreover, since $\{\xi_i\}_1^\infty$ is maximal, we have codim $(\sum_1^\infty \xi_i Q) < \infty$. Hence codim $(bQ) < \infty$ and thus $\bar{b}$ is a unit in $\bar{Q}$. Thus we have shown that $\bar{Q} = \{\bar{a}\bar{b}\colon \bar{a} \in \bar{R},\ \bar{b}$ unit of $\bar{Q}\}$. By an argument similar to that used in the proof of 4.3.5, it follows that $\bar{Q} = \{\bar{a}\bar{b}^{-1}\colon \bar{a}, \bar{b} \in \bar{R},\ \bar{b}$ unit of $\bar{Q}\}$. The proof of 4.2.14 can now be used to show R is a right order in Q.

BIBLIOGRAPHY

Amitsur, S.A.

[1] *Rings of quotients and Morita contexts*, J. Algebra 17 (1971), 273-298.

Barnes, B.A.

[1] *The Fredholm elements of a ring*, Canad. J. Math. 21 (1969), 84-95. MR 38 # 5823.

Beachy, J.A.

[1] *Bicommutators of cofaithful, fully divisible modules*, Canad. J. Math. 23 (1971), 202-213.

Cateforis, V.C.

[1] *Flat regular quotient rings*, Trans. Amer. Math. Soc. 138 (1969), 241-249. MR 39 # 259.

[2] *On regular self-injective rings*, Pacific J. Math. 30 (1969), 39-45. MR 40 # 1432.

[3] *Two-sided semisimple maximal quotient rings*, Trans. Amer. Math. Soc. 149 (1970), 339-349. MR 41 # 5424.

Cateforis, V.C.; Sandomierski, F.L.

[1] *On modules of singular submodule zero*, Canad. J. Math. 23 (1971), 345-354.

Chase, S.U.; Faith, C.

[1] *Quotient rings and direct products of full linear rings*, Math. Z. 88 (1965), 250-264. MR 31 # 2281.

Dlab, V.

[1] *Matrix representation of torsion-free rings*, Czech. Math. J. 19 (1969), 284-298. MR 39 # 6918.

Faith, C.

[1] *Lectures on injective modules and quotient rings*, Lecture Notes in Math., no. 49, Springer-Verlag, New York, 1967. MR 37 # 2791.

Faith, C.; Utumi, Y.

[1] *Intrinsic extensions of rings*, Pacific J. Math. 14 (1964), 505-512. MR 28 # 5077.

[2] *On noetherian prime rings*, Trans. Amer. Math. Soc. 114 (1965), 53-60. MR 30 # 3111.

Findlay, G.D.
[1] *Flat epimorphic extensions of rings,* Math. Z. 118 (1970), 281-288.

Findlay, G.D.; Lambek, J.
[1] *A generalized ring of quotients* I, Canad. Math. Bull. 1 (1958), 77-85. MR 20 # 888.
[2] *A generalized ring of quotients* II, ibid., 155-167.

Fort, J.
[1] *Sommes directes de sous-modules co-irréductibles d'un module,* Math. Z. 103 (1968), 363-388. MR 37 # 1400.

Fuller, K.R.
[1] *Double centralizers of injectives and projectives over Artinian rings,* Illinois J. Math. 14 (1970), 658-664.

Gabriel, P.
[1] *Des catégories abéliennes,* Bull. Soc. Math. France 90 (1962), 323-448. MR 38 # 1144.

Goldie, A.W.
[1] *The structure of prime rings under ascending chain conditions,* Proc. London Math. Soc. 8 (1958), 589-608. MR 21 # 1988.
[2] *Semi-prime rings with maximum condition,* Proc. London Math. Soc. 10 (1960), 201-220. MR 22 # 2627.

Gupta, R.N.
[1] *Self-injective quotient rings and injective quotient modules,* Osaka J. Math. 5 (1968), 69-87. MR 38 # 3294.

Harada, M.
[1] *The structure of rings whose quotient rings are primitive rings with minimal one-sided ideals,* J. Inst. Polytech. Osaka City Univ. Ser. A 11 (1960), 5-13. MR 23 # A170.

Hutchinson, J.J.
[1] *Intrinsic extensions of rings,* Pacific J. Math. 30 (1969), 669-677. MR 41 # 1806.
[2] *Quotient full linear rings,* Proc. Amer. Math. Soc. 28 (1971), 375-378.

Jacobson, N.
[1] *Structure of rings,* 2nd rev. ed., Amer. Math. Soc. Colloq. Publ., vol. 37, Amer. Math. Soc., Providence, R.I., (1964). MR 36 # 5158.

Johnson, R.E.
[1] *The extended centralizer of a ring over a module,* Proc. Amer. Math. Soc. 2 (1951), 891-895. MR 13, 618.
[2] *Structure theory of faithful rings II. Restricted rings,* Trans. Amer. Math. Soc. 84 (1957), 523-544. MR 18, 869.
[3] *Structure theory of faithful rings, III. Irreducible rings,* Proc. Amer. Math. Soc. 11 (1960), 710-717. MR 22 # 9510.
[4] *Quotient rings of rings with zero singular ideal,* Pacific J. Math. 11 (1961), 1385-1392. MR 26 # 1331.
[5] *Rings of finite rank,* Publ. Math. Debrecen 11 (1964), 284-287. MR 31 # 3454.
[6] *Remarks on a paper of Procesi,* J. Algebra 2 (1965), 38-41. MR 30 # 2040.

Johnson, R.E.; Wong, E.T.
[1] *Self-injective rings,* Canad. Math. Bull. 2 (1959), 167-173.MR 21 # 5652.
[2] *Quasi-injective modules and irreducible rings,* J. London Math. Soc. 36 (1961), 260-268. MR 24 # A1295.

Koh, K.; Luh, J.
[1] *On a finite dimensional quasi-simple module,* Proc. Amer. Math. Soc. 25 (1970), 801-807. MR 41 # 6897.

Koh, K.; Mewborn, A.C.
[1] *Prime rings with maximal annihilator and maximal complement right ideals,* Proc. Amer. Math. Soc. 16 (1965), 1073-1076. MR 36 # 2653.

Lambek, J.
[1] *On Utumi's ring of quotients,* Canad. J. Math. 15 (1963), 363-370. MR 26 # 5024.
[2] *Lectures on rings and modules,* Blaisdell, Toronto, London, and Waltham (Mass.), 1966. MR 34 # 5857.

Lesieur, L.; Croisot, R.
[1] *Sur les anneaux premier noethērian à gauche,* Ann. Sci. École, Norm. Sup. 76 (1959), 161-183. MR 22 # 54.

Michler, G.
[1] *Der Satz von Artin-Wedderburn und der Satz von Goldie,* Math. Ann. 163 (1966), 299-304. MR 34 # 1358.

Miyashita, Y.
[1] *On quasi-injective modules,* J. Fac. Sci. Hokkaido Univ. Ser. I 18 (1965), 158-187. MR 30 # 2044.

Morita, K.
[1] *Localizations in categories of modules. I,* Math. Z. 114 (1970), 121-144.
[2] *Localizations in categories of modules. III,* Math. Z. 119 (1971), 313-320.
[3] *Flat modules, injective modules and quotient rings,* Math. Z. 120 (1971), 25-40.

Osofsky, B.L.
[1] *Cyclic injective modules of full linear rings,* Proc. Amer. Math. Soc. 17 (1966), 247-253. MR 32 # 7604.
[2] *Homological dimension and cardinality,* Trans. Amer. Math. Soc. 151 (1970), 641-649.

Popescu, N.; Spircu, T.
[1] *Quelques observations sur les épimorphismes plats (à gauche) d'anneaux,* J. Algebra 16 (1970), 40-59.

Popescu, N.; Spulber, D.
[1] *Sur les quasi-ordres (à gauche) dans un anneau,* J. Algebra 17 (1971), 474-481.

Procesi, C.
[1] *Sopra un teorema die Goldie reguardante la structtura degli anelli primi con condizioni di massimo,* Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. 34 (1963), 372-377. MR 28 # 2130.

Procesi, C.; Small, L.W.
[1] *On a theorem of Goldie,* J. Algebra 2 (1965), 80-84. MR 31 # 3457.

Sandomierski, F.L.
[1] *Semisimple maximal quotient rings,* Trans. Amer. Math. Soc. 128 (1967), 112-120. MR 35 # 5473.
[2] *Some examples of right self-injective rings which are not left self-injective,* Proc. Amer. Math. Soc. 26 (1970), 244-245.

Silver, L.

[1] *Noncommutative localizations and applications,* J. Algebra 7 (1967), 44-76. MR 36 # 205.

Tachikawa, H.

[1] *Localization and Artinian quotient rings,* Math. Z. 119 (1971), 239-253.

Utumi, Y.

[1] *On quotient rings,* Osaka Math. J. 8 (1956), 1-18. MR 18, 7.

[2] *On continuous regular rings and semisimple self injective rings,* Canad. J. Math. 12 (1960), 597-605. MR 22 # 8032.

[3] *On continuous regular rings,* Canad. Math. Bull. 4 (1961), 63-69. MR 23 # A2436.

[4] *On rings of which any one-sided quotient rings are two-sided,* Proc. Amer. Math. Soc. 14 (1963), 141-147. MR 26 # 137.

[5] *On prime J-rings with uniform one-sided ideals,* Amer. J. Math. 85 (1963), 583-596. MR 30 # 4792.

[6] *On continuous rings and self injective rings,* Trans. Amer. Math. Soc. 118 (1965), 158-173. MR 30 # 4793.

[7] *On the continuity and self-injectivity of a complete regular ring,* Canad. J. Math. 18 (1966), 404-412. MR 36 # 6457.

Wolfson, K.G.

[1] *An ideal-theoretic characterization of the ring of all linear transformations,* Amer. J. Math. 75 (1953), 358-386. MR 14, 718.

ABSTRACT

The thesis is a study of right orders in a (left) full linear ring $Q = \mathrm{Hom}_D(V,V)$, V a right vector space over a division ring D. Chapter I, as well as summarizing known results needed in the sequel, outlines the method of attack. In particular, it justifies the hypothesis "suppose Q is a (Johnson) right quotient ring of R" which appears frequently in subsequent chapters. One of the principal results in Chapter II is that right orders in full linear rings of countable dimension must be prime rings whereas in the uncountable case this need not be so. In fact, results in Chapter II suggest that a complete description of a right order in Q may be rather difficult if dim Q_Q is uncountable (here dim M_R refers to the uniform dimension of a module M_R).

Chapter III is a study of intrinsic extensions of prime rings. This study is required by the condition that regular elements of a right order R in Q be units in Q, since this actually implies Q is left intrinsic over R if Q has infinite dimension. The principal result on intrinsic extensions says that if S is a prime ring with zero right singular ideal, but not an integral domain, and if S contains uniform right ideals then S is a right quotient ring of any prime ring over which it is right

intrinsic. This result has several interesting corollaries, for example, if dim Q_Q is countable then a right order R in Q must have Q also as a left quotient ring.

The main goal of Chapter IV is finding suitable conditions to ensure a ring R will have a full linear ring as its left-flat epimorphic hull. To this end, two conditions are introduced: condition (A) which requires closed right ideals of R to be essential extensions of finitely generated right ideals, and the existence of a "reducing pair" of elements. The latter means a pair (β,γ) for which βR, γR and $\beta^r + \gamma^r$ are large right ideals of R. Taken together, these two conditions on a ring R having Q as a right quotient ring imply that for each $x \in Q$ there exists $c \in R$ such that c has a right inverse in Q and $xc \in R$. Ample evidence is produced to show that reducing pairs for infinite dimensional rings R play a similar role to primeness for finite dimensional R, in so far as determining when R is a right order in Q. An earlier result of Chapter IV says that if R is a prime ring which satisfies (A) and has Q as a two-sided quotient ring then R is a right order in Q only if R + socle Q is, that is, in so far as a study of right orders in Q is concerned, we can suppose R contains the socle of Q.

The final chapter contains, among other things, two internal characterizations of a right order R in an infinite dimensional full linear ring. One of these says:

A ring R is a right order in a left full linear ring of right dimension $\aleph \geqslant \aleph_0$ if and only if the following conditions are satisfied.

(i) R is a (Johnson) irreducible ring containing uniform right ideals and dim $R_R = \aleph$.

(ii) The closed right ideals of R are right annihilator ideals and if B is a right annihilator ideal with dim B_R = dim R_R then B has the form $b^{\ell r}$ for some $b \in R$ with $b^r = 0$.

(iii) R possesses a reducing pair of elements.